A CRITERION FOR A CONNECTED DG ALGEBRA TO BE HOMOLOGICALLY SMOOTH

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Abstract. This paper gives some new results on cone length of DG modules and global dimension of connected DG algebras. Suppose that M is a DG module over a connected DG algebra A such that $H(M)$ is bounded below. It is proved that M admits a minimal semi-free resolution F_M whose DG free class is equal to its cone length and the grade of the graded $H(A)$-module $H(M)$ is smaller than or equals to the cone length of M. Moreover, if M either admits a minimal Eilenberg-Moore resolution or satisfies the condition that its cone length equals to the grade of the graded $H(A)$-module $H(M)$, then its cone length equals to the projective dimension of $H(M)$. An interesting result induced from this is that the global dimension of a connected algebra A is equal to the global dimension of $H(A)$ and the cone length of ${}_Ak$ if either of the latter two invariants is finite and equals to the depth of $H(A)$. The notion of global dimension for DG algebras is a generalization of the global dimension for graded algebras. The cone length of A as a DG A^e-module is an upper bound of the global dimension of A. Suppose that A is a connected DG algebra such that $H(A)$ is a left Noetherian graded algebra. It is proved in this paper that A is homologically smooth if and only if the global dimension of A is finite, which is also equivalent to either the finiteness of the cone length of either ${}_Ak$ or ${}_{A^e}A$ or the condition that $\mathrm{D^f}(A) = \mathrm{D^c}(A)$.

INTRODUCTION

Over the past two decades, the introduction and application of DG homological methods and techniques have been one of the main areas in homological algebra. People have done a lot of work to extend some important results in homological ring theory to DG context. In classical theory of homological algebra, it is well known that the regular property of a commutative noetherian local ring can be characterized by the finiteness of projective dimensions for all finitely generated modules. In DG homological algebra, homologically smooth DG algebras play a similar role as regular ring do in classical homological ring theory. Just as regular rings can be characterized by homological properties of their modules, homologically smooth DG algebras can be studied through their DG modules. The motivation of this paper is to get a similar characterization for homologically smooth DG algebras. Recently, many research works on homologically smooth DG algebras appear in literature. In [Sh1, Sh2], Shklyarov described a Serre duality functor on the category of perfect DG modules and developed a Riemann-Roch Theorem for DG algebras respectively. In Shklyarov's work, the base DG algebras that he considered are all homologically smooth DG algebras. In [HW], He-Wu introduced the concept of Koszul DG algebra. When the Koszul DG algebra is homologically smooth and Gorenstein, they obtained a DG version of the Koszul duality. The author and Wu [MW2] proved that any homologically smooth connected cochain DG algebra A is cohomologically unbounded unless A is quasi-isomorphic to the

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simple algebra k. And we prove that the Ext-algebra of a homologically smooth DG algebra A is Frobenius if and only if both $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(A)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(A^{op})$ admit Auslander-Reiten triangles. Besides these, it is well known that any Calabi-Yau DG algebra introduced by Ginzburg in [Gin] is a special kind of homologically smooth DG algebra by its definition. Calabi-Yau algebras can be seen as a non-commutative versions of coordinate rings of Calabi-Yau varieties. The study of their various properties has become a hot research topic now due to its profound geometric background.

A natural way to study an algebra is via the various homological invariants of the modules on them. There have been many different kinds of invariants on DG module since Appasov's work [Apa], where he defined and investigated homological dimensions of DG modules over DG algebras from both resolutional and functorial points of view. In [FJ], Frankild and Jørgensen introduced k-projective dimension and k-injective dimension for DG modules over a local chain DG algebra. Later, Yekutieli-Zhang [YZ] introduce projective dimension $\mathrm{proj.dim}_A M$ and flat dimension $\mathrm{flat.dim}_A M$ for a DG module M over a homologically bounded DG algebra A. Each of these homological invariants for DG modules can be seen as a generalization of the corresponding classical homological dimensions of modules over a ring. However, it seems that none of them can be used to define a finite global dimension of a DG algebra. Inspired from the definition of free class for differential modules over a commutative ring in [ABI], the invariant DG free class for semi-free DG modules was introduced in [MW3] by the author and Wu. By the existence of semi-free resolution, we define the cone length of a DG A-module as the least DG free classes of all its semi-free resolutions. This invariant of DG modules plays a similar role as projective dimension of modules does in homological ring theory. It is well known in homological ring theory that the projective dimension of a module over a local ring is equal to the length of its minimal projective resolution. In [MW3, Proposition 3.6], it was proved that if a DG A-module M admits a minimal Eilenberg-Moore resolution F, then $\mathrm{cl}_A M = \mathrm{DGfree.class}_A F$. In this paper, we improve this result by the following theorem (see Theorem 3.5).

Theorem A. Let M be a DG A-module such that $H(M)$ is bounded below. Then there exists a minimal semi-free resolution F_M of M such that

$$\mathrm{DGfree.class}_A F_M = \mathrm{cl}_A M.$$

Theorem A tells us a way to compute the cone length of a DG A-module. Another invariant, which is closely related to the cone length of DG A-module M, is the projective dimension of $H(M)$. Suppose that M is a DG module over a connected DG algebra A such that $H(M)$ is bounded. If M either admits a minimal Eilenberg-Moore resolution or satisfies the condition that its cone length equals to the grade of $H(M)$, then its cone length equals to the projective dimension of the graded $H(A)$-module $H(M)$ (see Proposition 3.6 and Proposition 3.9).

In [Jor2], Jørgensen put forward a question on how to define global dimension of DG algebras. As explained in [MW3], it is reasonable to some degree to define left (resp. right) global dimension of a connected DG algebra A to be the supremum of the set of the cone lengthes of all DG A-modules (resp. A^{op}-modules). In this paper, we further confirm this by the following theorem (see Theorem 5.5)

Theorem B. Let A be a connected cochain DG algebra such that $H(A)$ is a Noetherian graded algebra. Then the following are equivalent:

(a) A is homologically smooth;
(b) $\mathrm{cl}_A k < +\infty$;
(c) $\mathrm{cl}_{A^e} A < +\infty$;
(d) $\mathrm{D}^c(A) = \mathrm{D}^f(A)$;
(e) $l.\mathrm{Gl.dim} A < +\infty$.

Let A be a connected cochain DG algebra such that $H(A)$ is a left Noetherian graded algebra. Theorem B indicates that A is homologically smooth if and only if any cohomologically finite DG A-module is compact. By the existence of Eilenberg-moore resolution, every cohomologically finite DG module is compact if $\mathrm{gl.dim}H(A) < +\infty$. A natural question is whether the converse is right. Theorem B also indicates that this is generally not true (see Remark 6.3). The counter example (Example 6.1) in the last section also indicates that the cohomology algebra $H(A)$ of a homologically smooth DG algebra A may have infinite global dimension. In spite of this, we have the following three interesting propositions (see Proposition 4.4, Proposition 4.5 and Proposition 4.6).

Proposition A. Let A be a connected DG algebra. Then we have

$$\mathrm{cl}_A k = 1 \Leftrightarrow l.\mathrm{Gl.dim}A = 1 \Leftrightarrow \mathrm{gl.dim}H(A) = 1.$$

Proposition B Let A be a connected DG algebra with $\mathrm{gl.dim}H(A) = 2$. Then

$$l.\mathrm{Gl.dim}A = \mathrm{cl}_A k = 2.$$

Proposition C. Let A be a connected DG algebra. If either $\mathrm{cl}_A k$ or $\mathrm{gl.dim}H(A)$ is finite and equals to $\mathrm{depth}_{H(A)}H(A)$, then

$$l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k.$$

Proposition A gives a characterization of connected cochain DG algebras with global dimension 1. Suppose that A is a connected DG algebra such that $H(A)$ is an Artin-Schelter regular algebra. Then

$$l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k$$

by Proposition C. Note that Proposition B is not an immediate corollary of Proposition C (see Remark 4.7) and the converse of Proposition B is generally not true. The DG algebra A in Example 6.1 is a connected cochain DG algebra with $l.\mathrm{Gl.dim}A = \mathrm{cl}_A k = 2$ while $\mathrm{gl.dim}H(A) = +\infty$ (see Remark 6.3).

1. PRELIMINARIES

In this section, we review some basics on differential graded (DG for short) homological algebra, whose main main novelty is the study of the internal structure of a category of DG modules from a point of view inspired by classical homological algebra. There is some overlap here with the papers [MW1, MW2, FHT]. It is assumed that the reader is familiar with basics on the theory of triangulated categories and derived categories. If this is not the case, we refer to [Nee, Wei] for more details on them.

Throughout this paper, k is a fixed field. A k-algebra A is called $\mathbb{Z}$-graded if A has a k-vector space decomposition $A = \oplus_{i\in\mathbb{Z}} A^i$ such that

$$A^iA^j \subseteq A^{i+j}$$

for all $i, j \in \mathbb{Z}$. A $\mathbb{Z}$-graded algebra A is called positively graded, if $A^i = 0$, for any $i < 0$. If a positively graded algebra A satisfies the condition that $A^0 = k$, then it is called a **connected graded algebra**.

Let A be an N-graded k-algebra. A **left graded A-module** is a graded k-vector space M together with a k-linear map of degree zero

$$\begin{array}{ccc} A\otimes M & \longrightarrow & M \\ a\otimes m & \mapsto & am \end{array}$$

such that $a(bm) = (ab)m$, for all graded elements $a, b \in A$ and $m \in M$. A right graded modules over the N-graded k-algebra A is defined analogously.

Let M and N be two left graded modules over an N-graded k-algebra A. An A-linear map $f : M \to N$ of degree k is a linear map of degree k such that $f(am) = (-1)^{k|a|}af(m)$, for any graded elements $a \in A, m \in M$. Especially, an A-linear map of degree 0 between left graded modules is called a **morphism of left graded A-modules**.

Let A be a $\mathbb{Z}$-graded k-algebra. If there is a k-linear map $\partial_A : A \to A$ of degree 1 such that $\partial_A^2 = 0$ and

$$\partial(ab) = (\partial a)b + (-1)^{n|a|}a(\partial b)$$

for all graded elements $a, b \in A$, then A is called a **differential graded k-algebra**.

For any differential graded (DG for short) k-algebra A, its underlying graded algebra obtained by forgetting the differential of A is denoted by $A^{\#}$. If $A^{\#}$ is a connected graded algebra, then A is called a **connected DG algebra**. It is easy to check that $H(A)$ is a connected graded algebra if A is a connected DG algebra. We denote A^{op} as the **opposite DG algebra** of A, whose multiplication is defined as $a \cdot b = (-1)^{|a|\cdot|b|}ba$ for all graded elements a and b in A.

Remark 1.1. *For any connected DG algebra A, it has the following maximal DG ideal*

$$\mathfrak{m} : \cdots \to 0 \to A^1 \stackrel{\partial_1}{\to} A^2 \stackrel{\partial_2}{\to} \cdots \stackrel{\partial_{n-1}}{\to} A^n \stackrel{\partial_n}{\to} \cdots .$$

Obviously, the enveloping DG algebra $A^e = A \otimes A^{op}$ of A is also a connected DG algebra with $H(A^e) \cong H(A)^e$, and its maximal DG ideal is $\mathfrak{m} \otimes A^{op} + A \otimes \mathfrak{m}^{op}$.

Let A be a DG k-algebra. A **left DG module over** A (DG A-module for short) is a graded $A^{\#}$-module together with a linear k-map $\partial_M : M \to M$ of degree 1 satisfying the Leibniz rule:

$$\partial_M(am) = \partial_A(a)m + (-1)^{|a|}a\partial_M(m),$$

for all graded elements $a \in A$, $m \in M$.

Remark 1.2. *A right DG module over a DG k-algebra A is defined similarly. It is easy to check that any right DG modules over A can be identified with DG A^{op}-modules.*

For any DG A-module M and $i \in \mathbb{Z}$, the **i-th suspension** of M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)^j = M^{j+i}$. If $m \in M^l$, the corresponding element in $(\Sigma^i M)^{l-i}$ is denoted by $\Sigma^i m$. We have $a\Sigma^i m = (-1)^{|a|i}\Sigma^i(am)$ and $\partial_{\Sigma^i M}(\Sigma^i m) = (-1)^i\Sigma^i\partial_M(m)$, for any graded elements $a \in A, m \in M$.

An **A-homomorphism** $f : M \to N$ **of degree** i between DG A-modules M and N is a k-linear map of degree i such that

$$f(am) = (-1)^{i\cdot|a|}af(m), \text{ for all } a \in A, m \in M.$$

Denote $\mathrm{Hom}_A(M, N)$ as the graded vector space of all graded A-homomorphisms from M to N. This is a complex with them differential ∂_{Hom} defined by

$$\partial_{\mathrm{Hom}}(f) = \partial_N \circ f - (-1)^{|f|}f \circ \partial_M$$

for all $f \in \mathrm{Hom}_A(M, N)$. A **morphism of DG A-modules** from M to N is an A-homomorphism f of degree 0 such that $\partial_N \circ f = f \circ \partial_M$. The induced map $H(f)$ of f on the cohomologies is then a morphism of left graded $H(A)$-modules. If $H(f)$ is an isomorphism, then f is called a **quasi-isomorphism**, which is denoted as $f : M \stackrel{\simeq}{\to} N$.

Let f and g be two morphisms of DG A-modules between M and N. If there is an A-homomorphism $\sigma : M \to N$ of degree -1 such that $f - g = \partial_N \circ \sigma + \sigma \circ \partial_M$, then we say that f and g are **homotopic** to each other and we write $f \sim g$. A DG A-module M is called **homotopically trivial** if $\mathrm{id}_M \sim 0$. A morphism

$f : M \to N$ of DG A-modules is called a **homotopy equivalence** if there is a morphism $h : N \to M$ such that $f \circ h \sim \mathrm{id}_N$ and $h \circ f \sim \mathrm{id}_M$. And h is called a **homotopy inverse** of f. It is easy to check that any homotopy equivalence is a quasi-isomorphism.

A DG A-module P (resp. I) is called **K-projective** (resp. **K-injective**) if the functor $\mathrm{Hom}_A(P, -)$ (resp. $\mathrm{Hom}_A(-, I)$) preserves quasi-isomorphisms. And a DG A-module F is called K-flat if the functor $- \otimes_A F$ preserves quasi-isomorphisms.

A **K-projective resolution** (resp. **K-flat resolution**) of a DG A-module M is a quasi-isomorphism $\theta : P \to M$, where P is a K-projective (resp. K-flat) DG A-module. Similarly, a K**-injective resolution** of M is defined as a quasi-isomorphism $\eta : M \xrightarrow{\simeq} I$, where I is a K-injective DG A-module.

A DG A-module is called **DG free**, if it is isomorphic to a direct sum of suspensions of A (note it is not a free object in the category of DG modules). Let Y be a graded set, we denote $A^{(Y)}$ as the free DG module $\oplus_{y\in Y} Ae_y$, where $|e_y| = |y|$ and $\partial(e_y) = 0$. Let M be a DG A-module.

A subset E of M is called a **semi-basis** if it is a free basis of $M^{\#}$ over $A^{\#}$ and has a decomposition $E = \bigsqcup_{i\ge 0} E^i$ as a union of disjoint graded subsets E^i such that

$$\partial(E^0) = 0 \ \text{ and } \ \partial(E^u) \subseteq A(\bigsqcup_{i<u} E^i) \ \text{ for all } \ u > 0.$$

A DG A-module F is called **semi-free** if there is a sequence of DG submodules

$$0 = F_{-1} \subset F_0 \subseteq \cdots \subseteq F_n \subset \cdots$$

such that $F = \cup_{n\ge 0} F_n$ and that each $F_n/F_{n-1} = A \otimes V(n)$ is a DG free A-module. The differential of F can be decomposed as $\partial_F = d_0 + d_1 + \cdots$, where $d_0 = \partial_A \otimes \mathrm{Id}$ and each $d_i, i \ge 1$ is an A-linear map satisfying $d_i(V(l)) \subseteq A^{\#} \otimes V(l-i)$. It is easy to check that a DG A-module is semi-free if and only if it admits a semi-basis.

A **semi-free resolution** of a DG A-module M is a quasi-isomorphism

$$\varepsilon : F \to M,$$

where F is a semi-free DG A-module. Sometimes, we just say that F is a semi-free resolution of M. Semi-free resolutions play a similar important role in DG homological algebra as ordinary free resolutions do in homological ring theory.

It is well known that any DG A-module admits a semi-free resolution (cf. [FHT, Proposition 6.6]). Particularly, there are two important kinds of semi-free resolutions. One kind is called minimal semi-free resolutions. A semi-free resolution F of M is **minimal** if $\partial_F(F) \subset \mathfrak{m}F$. Assume that F is a minimal semi-free resolution of M, then both $\mathrm{Hom}_A(F, k)$ and $k \otimes_A F$ have vanishing differentials. The other kind is called Eilenberg-Moore resolutions. Assume that $\varepsilon : P \xrightarrow{\simeq} M$ is a semi-free resolution. Then P admits a semi-free filtration:

$$0 = P(-1) \subset P(0) \subseteq \cdots \subseteq P(i) \subseteq \cdots$$

such that $P = \cup_i P(i)$, and each $P(i)/P(i-1) = A \otimes V(i), i \ge 1$ is a DG free A-module. Set $F^{-i}P = P(i)$, then P admits a filtration:

$$0 = F^1P \subseteq F^0P \subseteq F^{-1}P \subseteq \cdots \subseteq F^{-i}P \subseteq \cdots .$$

This filtration induces a spectral sequence $(E_r, d_r)_{r\ge 0}$ beginning with

$$E_0^{i,j}(P) = (F^iP/F^{i+1}P)^{i+j} = (A \otimes V(-i))^{i+j}.$$

By the definition of semi-free A-module, $d_0 = \partial_A \otimes id$. It is easy to check that $E_1^{i,*}(P) \cong H(A) \otimes \Sigma^{-i}V(-i)$. Let $F^1M = 0$ and $F^iM = M, i \le 0$. We get a filtration of M:

$$0 = F^1M \subset F^0M = F^{-1}M = \cdots .$$

Obviously, $\varepsilon : P \to M$ preserves the filtration. Hence ε induces chain map between the corresponding spectral sequence. We get a complex of $H(A)$-modules

$$\cdots \to E_1^{i-1,*}(P) \to E_1^{i,*}(P) \to \cdots \to E_1^{0,*}(P) \to H(M) \to 0 \qquad (1).$$

If the complex (1) is exact, then (1) is a free resolution of $H(M)$. And we say that $\varepsilon : P \stackrel{\simeq}{\to} M$ is an **Eilenberg-Moore resolution** of M (cf. [FHT, §20]). Eilenberg-Moore resolutions are called distinguished resolutions in [Fe, GM, KM].

On the existence of minimal semi-free resolutions and Eilenberg-Moore resolutions, we have the following two remarks.

Remark 1.3. [MW1, Proposition 2.4] *Let M be a cohomologically bounded below DG A-module with $b = \inf\{j | H^j(M) \neq 0\}$. Then there is a minimal semi-free resolution F_M of M with $F_M^{\#} \cong \coprod_{i\ge b} \Sigma^{-i}(A^{\#})^{(\Lambda^i)}$, where each Λ^i is an index set.*

Remark 1.4. (Gugenheim-May) *For any DG A-module M, every free resolution*

$$\cdots \stackrel{\partial_{i+1}}{\to} H(A)\otimes V(i) \stackrel{\partial_i}{\to} \cdots \stackrel{\partial_2}{\to} H(A)\otimes V(1) \stackrel{\partial_1}{\to} H(A)\otimes V(0) \stackrel{\varepsilon}{\to} H(M) \to 0$$

of $H(M)$ can be realized as an E_1-term of some Eilenberg-Moore resolution of M.

The readers can see [FHT] (P. 279 - 280) for a proof of Remark 1.4. From the proof of this theorem, we see that the semi-basis of the Eilenberg-Moore resolution is in one to one correspondence with the free basis of the terms in the free resolution of $H(M)$.

For any given DG k-algebra A, let $\mathrm{C}(A)$ be the category of left DG A-modules and morphisms of left DG A-modules. The **derived category** of the Abelian category $\mathrm{C}(A)$ is denoted by $\mathrm{D}(A)$, which is constructed from $\mathrm{C}(A)$ by inverting quasi-isomorphisms. The right derived functor of Hom, is denoted by $R\mathrm{Hom}$, and the left derived functor of $\otimes$, is denoted by ${}^L\otimes$. They can be computed via K-projective, K-injective and K-flat resolutions of DG modules. It is easy to check that $\mathrm{Hom}_{\mathrm{D}(A)}(M,N) = H^0(R\mathrm{Hom}_A(M,N))$, for any objects M,N in $\mathrm{D}(A)$.

According to the definition of compact object (cf. [Kr1, Kr2]) in a triangulated category with arbitrary coproduct, a DG A-module is called **compact**, if the functor $\mathrm{Hom}_{\mathrm{D}(A)}(M,-)$ preserves all coproducts in $\mathrm{D}(A)$. By [Kel, Theorem 5.3], a DG A-module M is compact, if and only if it is in the smallest triangulated thick subcategory of $\mathrm{D}(A)$ containing ${}_AA$. To use the language of topologists, a DG A-module is compact if it can be built finitely from ${}_AA$, using suspensions and distinguished triangles.

Remark 1.5. *By the existence of the minimal semi-free resolution, it is easy to see that a cohomologically bounded below DG A-module M is compact if and only if $\dim_k H(k\,{}^L\otimes_A M) = \dim_k H(R\mathrm{Hom}_A(M,k))$ is finite.*

As the notation used in [Jor1, Jor2, MW1, MW2, Sch], we denote $\mathrm{D^c}(A)$ as the full triangulated subcategory of $\mathrm{D}(A)$ consisting of compact DG A-modules. Compact DG modules play the same role as finitely presented modules of finite projective dimension do in ring theory ([Jor2]). By [MW1, Proposition 3.3], a DG A-module M is called compact if and only if it admits a minimal semi-free resolution with a finite semi-basis. A DG k-algebra A is called **homologically smooth** (cf. [Sh1, Sh2, KS]), if A is a compact DG A^e-module.

Remark 1.6. [MW3, Corollary 4.1]*For any connected DG algebra A, ${}_Ak$ is compact if and only if A is homologically smooth, which is also equivalent to the condition that k_A is compact.*

A DG A-module M is called **cohomologically finite** (resp. **cohomologically bounded**) if $H(M)$ is a finitely generated (resp. bounded) $H(A)$-module. The

full triangulated subcategory of $\mathrm{D}(A)$ consisting of cohomologically finite DG A-module and cohomologically bounded DG A-modules are denoted by $\mathrm{D}^f(A)$ and $\mathrm{D}^b(A)$ respectively. Finally, a DG A-module M is called **cohomologically locally finite**, if each $H^i(M)$ is finite dimensional. The full subcategory of $\mathrm{D}(A)$ consisting of cohomologically locally finite DG A-modules is denoted by $\mathrm{D}_{\mathrm{lf}}(A)$.

2. BASIC LEMMAS ON DG MODULES

For the rest of the paper, we fix A as a connected DG algebra over a field k if no special assumption is emphasized. In this section, we give some fundamental lemmas and propositions on DG A-modules.

Lemma 2.1. [FHT, Remark 20.1] *Any bounded below projective graded module over a connected graded algebra is a free graded module.*

Since any DG A-module is a graded $A^{\#}$-module by forgetting its differential, we can easily get the following lemma by the graded Nakayama Lemma.

Lemma 2.2. (DG Nakayama Lemma) *Let M be a bounded below DG A-module. If L is a DG A-submodule of M such that $L+\mathfrak{m}M=M$, then $L=M$.*

Lemma 2.3. *Let F be a bounded below DG A-module such that $\partial_F(F)\subseteq mF$ and $F^{\#}$ is a projective $A^{\#}$-module. If a DG morphism $\alpha: F\to F$ is homotopic to the identity morphism id_F, then α is an isomorphism.*

Proof. Since $\alpha\simeq \mathrm{id}_F$, there is a homotopy map $h: F\to F$ such that

$$\alpha-\mathrm{id}_F=h\circ\partial_F+\partial_F\circ h.$$

Let $\overline{F}=k\otimes_A F, \overline{\alpha}=k\otimes_A \alpha$ and $\overline{h}=k\otimes_A h$. Since $\partial_F(F)=\mathfrak{m}F$, we have $\overline{\alpha}=\mathrm{id}_{\overline{F}}+\overline{h}\circ\partial_{\overline{F}}+\partial_{\overline{F}}\circ\overline{h}=\mathrm{id}_{\overline{F}}$. Hence $F=\mathrm{im}(\alpha)+\mathfrak{m}F$. By Lemma 2.2, we have $\mathrm{im}(\alpha)=F$. Since $F^{\#}$ is a projective $A^{\#}$-module, the short exact sequence

$$0\to\ker(\alpha)\to F\stackrel{\alpha}{\to}F\to 0$$

is linearly split. Acting the functor $k\otimes_A -$ on this linearly split short exact sequence, gives a short exact sequence

$$0\to k\otimes_A\ker(\alpha)\to\overline{F}\stackrel{\overline{\alpha}}{\to}\overline{F}\to 0$$

of graded k-verctor spaces. Since $\overline{\alpha}$ is a monomorphism, we have

$$\ker(\alpha)/\mathfrak{m}\ker(\alpha)=k\otimes_A\ker(\alpha)=0.$$

Suppose that $\ker(\alpha)\neq 0$, then $\ker(\alpha)$ is a bounded below DG A-module since it is a DG A-submodule of F. This implies that $\ker(\alpha)\neq\mathfrak{m}\ker(\alpha)$. It contradicts with $\ker(\alpha)/\mathfrak{m}\ker(\alpha)=0$. Hence $\ker(\alpha)=0$. $\square$

Lemma 2.4. *Let M be a DG A-module. Then M is a homotopically trivial if and only if $H(\mathrm{Hom}_A(M,M))=0$.*

Proof. If M is homotopically trivial, then there is an A-homomorphism $\sigma: M\to M$ of degree -1 such that $\mathrm{id}_M=\partial_M\circ\sigma+\sigma\circ\partial_M$. For any cocycle $f\in\mathrm{Hom}_A(M,M)$, we have $\partial_{\mathrm{Hom}}(f)=\partial_M\circ f-(-1)^{|f|}f\circ\partial_M$ =0. Hence $\partial_M\circ f=(-1)^{|f|}f\circ\partial_M$. Then

$$\begin{aligned}\partial_{\mathrm{Hom}}(f\circ\sigma)&=f\circ\sigma\circ\partial_M-(-1)^{|f|-1}\partial_M\circ f\circ\sigma\\&=f\circ(\mathrm{id}_M-\partial_M\circ\sigma)+f\circ\partial_M\circ\sigma\\&=f.\end{aligned}$$

Thus $H(\mathrm{Hom}_A(M,M))=0$.

Conversely, suppose that $H(\mathrm{Hom}_A(M,M))=0$, we need to prove that M is homotopically trivial. Since $\partial_{\mathrm{Hom}}(\mathrm{id}_M)=\partial_M\circ \mathrm{id}_M-\mathrm{id}_M\circ\partial_M=0$, there is $\sigma\in \mathrm{Hom}_A(M,M)$ of degree -1 such that $\mathrm{id}_M=\partial_{\mathrm{Hom}}(\sigma)=\partial_M\circ\sigma+\sigma\circ\partial_M$. Therefore, M is homotopically trivial. $\square$

Lemma 2.5. *Let $f: F_1\to F_2$ be a DG morphism between two K-projective DG A-modules. Then f is a quasi-isomorphism if and only if f is a homotopy equivalence.*

Proof. If f is a homotopy equivalence, then there is a DG morphism $g: F_2\to F_1$ such that $g\circ f\sim \mathrm{id}_{F_1}$ and $f\circ g\sim\mathrm{id}_{F_2}$. There are two A-homomorphisms

$$\sigma_1: F_1\to F_1 \quad\text{and}\quad \sigma_2: F_2\to F_2$$

of degree -1 such that $g\circ f-\mathrm{id}_{F_1}=\partial_{F_1}\circ\sigma_1+\sigma_1\circ\partial_{F_1}=\partial_{\mathrm{Hom}_A(F_1,F_1)}(\sigma_1)$ and

$$f\circ g-\mathrm{id}_{F_2}=\partial_{F_2}\circ\sigma_2+\sigma_2\circ\partial_{F_2}=\partial_{\mathrm{Hom}_A(F_2,F_2)}(\sigma_2).$$

This implies that $H(f)\circ H(g)=H(f\circ g)=H(\mathrm{id}_{F_1})=\mathrm{id}_{H(\mathrm{Hom}_A(F_1,F_1))}$ and

$$H(g)\circ H(f)=H(g\circ f)=H(\mathrm{id}_{F_2})=\mathrm{id}_{H(\mathrm{Hom}_A(F_2,F_2))}.$$

Hence $H(f)$ is an isomorphism of graded $H(A)$-modules. So f is a quasi-isomorphism.

Conversely, if f is a quasi-isomorphism, then

$$\mathrm{Hom}_A(F_2,f):\mathrm{Hom}_A(F_2,F_1)\to\mathrm{Hom}_A(F_2,F_2)$$

is a quasi-isomorphism since F_2 is a K-projective DG A-module. For $[\mathrm{id}_{F_2}]\in H(\mathrm{Hom}_A(F_2,F_2))$, there is DG morphism $g: F_2\to F_1$ such that

$$H(\mathrm{Hom}_A(F_2,f))[g]=[f\circ g]=[\mathrm{id}_{F_2}].$$

So there is an A-homomorphisms $\sigma_2: F_2\to F_2$ satisfying

$$f\circ g-\mathrm{id}_{F_2}=\partial_{\mathrm{Hom}_A(F_2,F_2)}(\sigma_2)=\partial_{F_2}\circ\sigma_2+\sigma_2\circ\partial_{F_2}$$

and $H(f)\circ H(g)=[f\circ g]=[\mathrm{id}_{F_2}]=\mathrm{id}_{H(F_2)}$. Thus $f\circ g\sim\mathrm{id}_{F_2}$ and g is also a quasi-isomorphism, since $H(f)$ is an isomorphism. Since F_1 is K-projective,

$$\mathrm{Hom}_A(F_1,g):\mathrm{Hom}_A(F_1,F_2)\to\mathrm{Hom}_A(F_1,F_1)$$

is a quasi-isomorphism. For $[\mathrm{id}_{F_1}]\in H(\mathrm{Hom}_A(F_1,F_1))$, there is DG morphism $f': F_1\to F_2$ such that $H(\mathrm{Hom}_A(F_1,g))[f']=[g\circ f']=[\mathrm{id}_{F_1}]$. Thus there is an A-homomorphisms $\sigma_1: F_1\to F_1$ satisfying

$$g\circ f'-\mathrm{id}_{F_1}=\partial_{\mathrm{Hom}_A(F_1,F_1)}(\sigma_1)=\partial_{F_1}\circ\sigma_1+\sigma_1\circ\partial_{F_1}.$$

So $g\circ f'\sim\mathrm{id}_{F_1}$. Hence we have $f\sim f\circ g\circ f'\sim \mathrm{id}_{F_2}\circ f'=f'$ and then $g\circ f\sim g\circ f'\sim\mathrm{id}_{F_1}$. Therefore, f is a homotopy equivalence. $\square$

Lemma 2.6. [FHT, Proposition 6.4] *Suppose that F is a semi-free DG A-module and $\eta: M\to N$ is a quasi-isomorphism. Then $\mathrm{Hom}_A(F,\eta)$ is a quasi-isomorphism. Hence any semi-free DG A-module is K-projective.*

Lemma 2.7. [FHT, Proposition 6.6] *Any DG A-module M admits a semi-free resolution $f: F\overset{\simeq}{\twoheadrightarrow} M$. And if $g: G\overset{\simeq}{\twoheadrightarrow} M$ is a second semi-free resolution then there is a homotopy equivalence $h: G\to F$ such that $g\sim f\circ h$.*

Proposition 2.8. *Let F be a DG free A-module such that $F\cong G\oplus Q$ as a DG A-module, where G and Q are two DG A-modules. If $H(G)$ is bounded below, then G is a DG free A-module.*

Proof. Let $\{e_i|i \in I\}$ be a DG free basis of the DG free A-module F. It is easy to check that $H(F)$ is a free graded $H(A)$-module with
$$H(F) \cong H(G) \oplus H(Q) \cong \bigoplus_{i\in I} H(A)[e_i].$$
Hence $H(G)$ is a projective graded $H(A)$-module. By the assumption that $H(G)$ is bounded below and Lemma 2.1 , $H(F)$ is a free graded $H(A)$-module. Let
$$H(G) = \bigoplus_{j\in J} H(A)[f_j],$$
where each f_j is a cocycle element of G. Let L be a DG free A-module with a DG free basis $\{x_j|j \in J\}$. Define a morphism of DG A-modules $\varepsilon : L \to G$ by $\varepsilon(x_j) = f_j$, for any $j \in J$. It is easy to see that ε is a quasi-isomorphism. Since G is a direct summand of a DG free A-module, G is a semi-projective DG A-module. By Lemma 2.5, ε is a homotopy equivalence. Let $\theta : G \to L$ be the homotopy inverse of ε. We have $\varepsilon \circ \theta \simeq \mathrm{id}_G$ and $\theta \circ \varepsilon \simeq \mathrm{id}_L$. Since any DG free A-module is minimal, we have $\partial_L(L) \subseteq \mathfrak{m}L$ and $\partial_F(F) \subseteq \mathfrak{m}F$. Since G is a direct summand of F, $G^{\#}$ is projective and $\partial_G(G) \subseteq \mathfrak{m}G$. By Lemma 2.3, $\varepsilon \circ \theta$ and $\theta \circ \varepsilon$ are isomorphisms. This implies that ε is a bijective. Hence $L \cong G$ and G is a DG free A-module.

□

Proposition 2.9. *Let F be a semi-free DG A-module such that $H(F)$ is bounded below. Then there is a minimal semi-free resolution G of F and a homotopically trivial DG A-module Q such that $F \cong G \oplus Q$ as a DG A-module.*

Proof. By Remark 1.3, F admits a minimal semi-free resolution $g : G \to F$ with $\inf\{i|G^i \neq 0\} = \inf\{i|H^i(F) \neq 0\}$. Since F can be seen as a semi-free resolution of itself, there is a homotopy equivalence $h : G \to F$ such that $\mathrm{id}_F \circ h \sim g$ by Lemma 2.7. Let $f : F \to G$ be the homotopy inverse of h. Then $f \circ h \sim \mathrm{id}_G$. Hence there is an A-linear homomorphism $\sigma : G \to G$ of degree -1 such that $f \circ h - \mathrm{id}_G = \partial_G \circ \sigma + \sigma \circ \partial_G$. Since $\partial_G(G) \subseteq \mathfrak{m}G$ and σ is A-linear, we have $f \circ h - \mathrm{id}_G \subseteq \mathfrak{m}G$. Hence $\overline{f \circ h} = k \otimes_A (f \circ h)$ is the identity map of $\overline{G} = G/\mathfrak{m}G = k \otimes_A G$. Acting on the exact sequence
$$G \xrightarrow{f\circ h} G \longrightarrow \mathrm{coker}(f \circ h) \longrightarrow 0$$
by $k \otimes_A -$ gives a new exact sequence
$$\overline{G} \xrightarrow{\overline{f\circ h}} \overline{G} \longrightarrow \overline{\mathrm{coker}(f \circ h)} \longrightarrow 0.$$
This implies that $\overline{\mathrm{coker}(f \circ h)} = 0$. Hence $\mathrm{coker}(f \circ h) = \mathfrak{m} \cdot \mathrm{coker}(f \circ h)$. If $\mathrm{coker}(f \circ h) = G/\mathrm{im}(f \circ h)$ is not zero, then it is bounded below since G is bounded below. Let $v = \inf\{i|(\mathrm{coker}(f \circ h))^i \neq 0\}$. Since $\mathfrak{m}$ is concentrated in degrees ≥ 1, $\mathfrak{m} \cdot \mathrm{coker}(f \circ h)$ is concentrated in degrees $\geq v + 1$. This contradicts with $\mathrm{coker}(f \circ h) = \mathfrak{m} \cdot \mathrm{coker}(f \circ h)$. Therefore, $\mathrm{coker}(f \circ h) = 0$ and $f \circ h$ is surjective. We have the following linearly split short exact sequence
$$0 \longrightarrow \ker(f \circ h) \longrightarrow G \xrightarrow{f\circ h} G \longrightarrow 0 \qquad (1).$$
Acting on (1) by $k \otimes_A -$ gives a new short exact sequence
$$0 \longrightarrow \overline{\ker(f \circ h)} \longrightarrow \overline{G} \xrightarrow{\overline{f\circ h}} \overline{G} \longrightarrow 0.$$
This implies that $\overline{\ker(f \circ h)} = 0$ since $\overline{f \circ h}$ is the identity map. Hence $\ker(f \circ h) = \mathfrak{m} \cdot \ker(f \circ h)$. If $\ker(f \circ h)$ is not zero, then it is bounded below since it is a DG A-submodule of G. Let $u = \inf\{i|(\ker(f \circ h))^i \neq 0\}$. Then $\mathfrak{m} \cdot \ker(f \circ h)$ is concentrated in degrees $\geq u + 1$. This contradicts with $\ker(f \circ h) = \mathfrak{m} \cdot \ker(f \circ h)$. Thus $\ker(f \circ h) = 0$ and $f \circ h$ is an isomorphism. Let $\theta : G \to G$ be the inverse of

$f \circ h$. Then $\theta \circ f \circ h = \mathrm{id}_G$. This implies that h is a monomorphism and the short exact sequence

$$0 \longrightarrow G \xrightarrow{h} F \longrightarrow \mathrm{coker}(h) \longrightarrow 0$$

is split. Hence $F \cong G \oplus \mathrm{coker}(h)$ as a DG A-module. Since $H(h)$ is an isomorphism, $\mathrm{coker}(h)$ is quasi-trivial. By Lemma 2.6, both $\mathrm{Hom}_A(F, \mathrm{coker}(h))$ and $\mathrm{Hom}_A(G, \mathrm{coker}(h))$ are quasi-trivial. Hence $H(\mathrm{Hom}_A(\mathrm{coker}(h), \mathrm{coker}(h))) = 0$, since $\mathrm{Hom}_A(F, \mathrm{coker}(h)) \cong \mathrm{Hom}_A(G, \mathrm{coker}(h)) \oplus \mathrm{Hom}_A(\mathrm{coker}(h), \mathrm{coker}(h))$. By Lemma 2.4, $\mathrm{coker}(h)$ is homotopically trivial.

□

Lemma 2.10. [KM, Lemma III.1.2] *Let $f : F \to G$ be morphism of DG A-modules between two semi-free A-modules. Given a semi-free filtration of G*

$$0 \subseteq G(0) \subseteq G(1) \subseteq \cdots \subseteq G(i) \subseteq \cdots ,$$

there is a semi-free filtration $0 \subseteq F(0) \subseteq F(1) \subseteq \cdots \subseteq F(i) \subseteq \cdots$ of F such that $f(F(i)) \subseteq G(i)$, for any $i \geq 0$.

Let $f : F \to G$ be a morphism of DG A-modules, where F is a semi-free DG A-module with a strictly increasing semi-free filtration

$$0 = F(-1) \subset F(2) \subset \cdots \subset F(n) \subset \cdots ,$$

and G is an Eilenberg-Moore resolution of itself constructed from a free resolution of $H(G)$

$$\cdots \xrightarrow{\partial_{n+1}} H(A) \otimes W(n) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} H(A) \otimes W(0) \xrightarrow{\varepsilon} H(G) \to 0 \qquad (2).$$

We know that G admits a strictly increasing semi-free filtration

$$0 = G(-1) \subset G(0) \subset \cdots \subset G(n) \subset \cdots$$

such that each $G(i)/G(i-1) = A \otimes \Sigma^i W(i), i \geq 0$.

Proposition 2.11. *Let $f : F \to G$ be a morphism DG A-modules satisfying the conditions above. Then f is homotopic to a DG morphism $g : F \to G$ such that $g(F(i)) \subseteq G(i)$, for any $i \geq 0$.*

Proof. For each $i \geq 0$, $F(i)/F(i-1) = A \otimes V(i)$ is a DG free A-module on a cocycle basis $\{e_{i_j} | j \in I_i\}$. Since $G^\# = \oplus_{i \geq 0}(A^\# \otimes \Sigma^i W(i))$, we may write

$$f(e_{0_j}) = \sum_{i=0}^{n_j} f_i(e_{0_j}),$$

where each $f_i(e_{0_j}) \in A^\# \otimes \Sigma^i W(i)$.

If $n_j > 0$, we have $d_0(f_{n_j}(e_{0_j})) = 0$ and $d_1(f_{n_j}(e_{0_j})) = -d_0(f_{n_j-1}(e_{0_j}))$ since $d_G \circ f(e_{0_j}) = f \circ d_F(e_{0_j}) = 0$. So $x_i = [\Sigma^{-n_i} f_{n_i}(e_{0_j})] \in H(A) \otimes W(n_i)$ and $\partial_{n_i}[x_i] = 0$. By the exactness of (2), there exists $y_i \in H(A) \otimes W(n_i + 1)$ such that $\partial_{n_i+1}(y_i) = x_i$. Let u_i be a cocycle in $(A \otimes W(n_i + 1), \partial_A \otimes \mathrm{id})$ representing y_i. Hence there exists $v_i \in (A \otimes W(n_i), \partial_A \otimes \mathrm{id})$ such that

$$d_1(\Sigma^{n_i+1} u_i) + d_0(\Sigma^{n_i} v_i) = f_{n_j}(e_{0_j}).$$

We define a map $H : F \to G$ such that $H|_{A \otimes V_r} = 0$ for $r \neq 0$ and $H(e_{0_j}) = \Sigma^{n_i+1} u_i + \Sigma^{n_i} v_i$. Take $f' = f - H \circ d_F - d_G \circ H$. Surely, $f \sim f'$ and $f'(e_{0_j}) \in G(n_j - 1), j \in I_0$. Iterating the procedure above, we can construct a DG morphism $g \sim f$ such that $g(V(0)) \subseteq G(0) = A \otimes W(0)$.

Suppose inductively that $f(V(i)) \subseteq G(i)$, for $i = 0, 1, \cdots , r-1$. We denote

$$f(e_{r_j}) = \sum_{i=0}^{t_j} f_i(e_{r_j}),$$

where $f_i(e_{r_j}) \in A \otimes \Sigma^i W(i)$. If $t_j > r$, then $d_0 \circ f_{t_j}(e_{r_j}) = 0$ and $d_1 \circ f_{t_j}(e_{r_j}) = -d_0 \circ f_{t_j-1}(e_{r_j})$. Using the same method as above, we can construct f' such that $f \sim f'$ and $f'(V(r)) \subseteq G(r)$.

By the induction above, f is homotopic to a morphism g of DG A-modules such that $g(F(r)) \subseteq G(r)$, for any $r \geq 0$. □

3. INVARIANTS OF DG MODULES

The terminology 'class' in group theory is used to measure the shortest length of a filtration with sub-quotients of certain type. Carlsson [Car] introduced 'free class' for solvable free differential graded modules over a graded polynomial ring. Recently, Avramov, Buchweitz and Iyengar [ABI] introduce free class, projective class and flat class for differential modules over a commutative ring. Inspired from them, we define the DG free class of a semi-free DG module F as the shortest length of all strictly increasing semi-free filtrations of F.

Definition 3.1. *Let F be a semi-free DG A-module. A semi-free filtration of F*

$$0 = F(-1) \subseteq F(0) \subseteq \cdots \subseteq F(n) \subseteq \cdots$$

is called strictly increasing, if $F(i-1) \neq F(i)$ when $F(i-1) \neq F, i \geq 0$. If there is some n such that $F(n) = F$ and $F(n-1) \neq F$, then we say that this strictly increasing semi-free filtration has length n. If no such integer exists, then we say the length is $+\infty$.

Definition 3.2. *Let F be a semi-free DG A-module. The DG free class of F is defined to be the number*

$$\inf\{n \in \mathbb{N} \cup \{0\} \,|\, F \textit{ admits a strictly increasing semi-free filtration of length } n\}.$$

We denote it as $\mathrm{DGfree\ class}_A F$.

Lemma 3.3. *Let F be a semi-free DG A-module and let F' be a semi-free DG submodule of F such that $F/F' = A \otimes V$ is DG free on a set of cocycles. Then there exists a DG morphism $f : A \otimes \Sigma^{-1} V \to F'$ such that $F = \mathrm{cone}(f)$.*

Proof. Let $\{e_i | i \in I\}$ be a basis of V. Define a DG morphism $f : A \otimes \Sigma^{-1} V \to F'$ by $f(\Sigma^{-1} e_i) = \partial_F(e_i)$. It is easy to check that $\partial_{\mathrm{cone}(f)}(e_i) = f(\Sigma^{-1} e_i) = \partial_F(e_i)$. Hence $F = \mathrm{cone(f)}$. □

In rational homotopy theory, cone length of a topological space X is defined to be the least m such that X has the homotopy type of an m-cone. It is a useful invariant in the evaluation of Lusternik-Schnirelmann category, which is an important invariant of homotopy type. In this paper, we define the cone length of a DG A-module as the infimum of the set of DG free classes of all its semi-free resolutions.

Definition 3.4. *Let M be a DG A-module. The cone length of M is defined to be the number*

$$\mathrm{cl}_A M = \inf\{\, \mathrm{DGfree\ class}_A F \,|\, F \xrightarrow{\simeq} M \textit{ is a semi-free resolution of } M\}.$$

Note that $\mathrm{cl}_A M$ may be $+\infty$. We call this invariant 'cone length' because semi-free DG A-modules can be constructed by iterative cone constructions from DG free A-modules (see Lemma 3.3) and any DG A-module admits semi-free resolutions. Cone length of a DG A-module plays a similar role in DG homological algebra as projective dimension of a module over a ring does in classic homological ring theory.

Theorem 3.5. *Let M be a DG A-module such that $H(M)$ is bounded below. Then there exists a minimal semi-free resolution F_M of M such that*

$$\mathrm{DGfree.class}_A F_M = \mathrm{cl}_A M.$$

Proof. By Remark 1.3, M admits a minimal semi-free resolution F_M. If $\mathrm{cl}_A M = +\infty$, then $\mathrm{DGfree.class}_A F_M = +\infty$ since $\mathrm{cl}_A M \le \mathrm{DGfree.class}_A F_M$ by the definition of cone length.

Therefore, it suffices to prove the proposition when $\mathrm{cl}_A M$ is finite. Suppose that $\mathrm{cl}_A M = n$, for some $n \in \mathbb{N}$. Then there is a semi-free resolution P of M such that $\mathrm{DGfree.class}_A P = n$. If P is minimal, then we are done. Suppose that P is not a minimal semi-free DG A-module. By Proposition 2.9, there is a minimal semi-free resolution G of P such that $P \cong G \oplus Q$. The composition map $G \overset{\simeq}{\to} P \overset{\simeq}{\to} M$ is a minimal semi-free resolution of M. Let $F = G \oplus Q$. Since $\mathrm{DGfree.class}_A P = n$ and $F \cong P$ as a DG A-module, F admits a semi-free filtration

$$0 = F(-1) \subset F(0) \subset F(1) \subset \cdots \subset F(n) = F.$$

Let $i : Q \to F$ and $\pi : F \to G$ be the canonical inclusion map and the projection map respectively. Since $F = G \oplus Q$ as a DG A-module, there is a split short exact sequence $0 \to Q \overset{i}{\to} F \overset{\pi}{\to} G \to 0$ of DG A-modules. Since π is surjective, we have a filtration

$$0 \subseteq \pi(F(0)) \subseteq \pi(F(1)) \subseteq \cdots \subseteq \pi(F(n-1)) \subseteq \pi(F(n)) = G$$

of G. We want to prove that this is a semi-free filtration of G.

Let $F(i)/F(i-1) = A \otimes V_i$, $i = 0, 1, 2, \cdots, n$. Each V_i is a subspace of F and so is $V_{\le t} = \bigoplus_{i=0}^{t} V_i$, $t = 0, 1, 2, \cdots, n$. The free graded $A^{\#}$-module $F(j)^{\#} = A^{\#} \otimes V_{\le j}$, $j = 0, 1, \cdots, n$. Since π is k-linear, we have $V_{\le t} \cong \pi(V_{\le t}) \oplus \ker(\pi|_{V_{\le t}})$. It is straightforward to verify that $(\pi(F(i)))^{\#} = A^{\#} \otimes \pi(V_{\le i})$ since π is A-linear.

For any $v_0 \in V_0$, we have $\partial_G(\pi(v_0)) = \pi \circ \partial_F(v_0) = 0$. This implies that $\pi(F(0)) = A \otimes \pi(V_0)$ is a DG free A-modules. It remains to prove that each $\pi(F(i))/\pi(F(i-1))$ is a DG free A-module, $i = 1, 2, \cdots, n$. Since $\pi(V_{\le i-1})$ is a subspace of $\pi(V_{\le i})$, we have

$$(\pi(F(i))/\pi(F(i-1)))^{\#} = \frac{A^{\#} \otimes \pi(V_{\le i})}{A^{\#} \otimes \pi(V_{\le i-1})} = A^{\#} \otimes (\pi(V_{\le i})/\pi(V_{\le i-1})).$$

For any $v \in V_{\le i}$, we have $\partial_G(\pi(v)) = \pi \circ \partial_F(v) \in \pi(F(i-1))$ since $\partial_F(V_i) \subseteq F(i-1)$. Hence each $\pi(F(i))/\pi(F(i-1))$ is a DG free A-module, $i = 1, 2, \cdots, n$. Therefore, the filtration of G

$$0 \subseteq \pi(F(0)) \subseteq \pi(F(1)) \subseteq \cdots \subseteq \pi(F(n-1)) \subseteq \pi(F(n)) = G$$

is a semi-free filtration of G. By the definition of DG free class, $\mathrm{DGfree.class}_A G \le n$. On the other hand, $\mathrm{DGfree.class}_A G \ge \mathrm{cl}_A M = n$ by the definition of cone length. So $\mathrm{DGfree.class}_A G = n$.

□

For any DG A-module M, it is easy to check that $\mathrm{cl}_A M \le \mathrm{pd}_{H(A)} H(M)$ by the existence of Eilenberg-Moore resolution. The following two propositions tell us some sufficient conditions for $\mathrm{cl}_A M = \mathrm{pd}_{H(A)} H(M)$.

Proposition 3.6. *Let M be DG A-module such that $H(M)$ is bounded below . If M admits a minimal Eilenberg-Moore resolution. Then $\mathrm{cl}_A M = \mathrm{pd}_{H(A)} H(M)$.*

Proof. If $\mathrm{cl}_A M = +\infty$, then $\mathrm{pd}_{H(A)} H(M) = +\infty$ since $\mathrm{cl}_A M \le \mathrm{pd}_{H(A)} H(M)$. We need to prove that $\mathrm{pd}_{H(A)} H(M) = n$, if $\mathrm{cl}_A M = n$, for some integer n. It suffices to prove that $\mathrm{pd}_{H(A)} H(M) \le n$.

If $\mathrm{pd}_{H(A)} H(M) = l > n$, then $H(M)$ admits a minimal free resolution

$$0 \to H(A) \otimes W(l) \overset{\partial_l}{\to} \cdots \overset{\partial_2}{\to} H(A) \otimes W(1) \overset{\partial_1}{\to} H(A) \otimes W(0) \overset{\varepsilon}{\to} H(M) \to 0 \quad (3).$$

By assumption, M admits a minimal Eilenberg-Moore resolution F constructed from (3). Then F admits a strictly increasing semi-free filtration

$$0 = F(-1) \subset F(0) \subset F(1) \subset \cdots \subset F(l-1) \subset F(l) = F,$$

such that $F(i)/F(i-1) = A \otimes \Sigma^i W(i), 0 \leq i \leq 1$.

On the other hand, DGfree $\text{class}_A F = n$ by Theorem 3.5. Hence F admits another semi-free filtration:

$$0 \subset G(0) \subset G(1) \subset \cdots \subset G(n) = F,$$

where $G(i)/G(i-1) = A \otimes V(i), i \geq 1$.

By Proposition 2.11, the identity morphism $\text{id}_F : F \to F$ is homotopic to a morphism of DG A-modules $g : F \to F$, such that $g(G(i)) \subseteq F(i), 0 \leq i \leq n$. Since $g \sim \text{id}_F$ and F is a semi-free A-module, g is a homotopy equivalence. Therefore,

$$\text{id}_k \otimes_A g : k \otimes_A F \to k \otimes_A F$$

is a quasi-isomorphism. But $\text{im}(g) \subseteq F(n)$, so

$$\text{im}(\text{id}_k \otimes_A g) \subseteq \bigoplus_{i=0}^{n} \Sigma^i W(i) \subset \bigoplus_{i=0}^{l} \Sigma^i W(i).$$

From this, we conclude that $H(\text{id}_k \otimes_A g) = \text{id}_k \otimes_A g$ is not an isomorphism. This contradicts with the fact that $\text{id}_k \otimes_A g$ is a quasi-isomorphism. Thus $l \leq n$. $\square$

By Remark 1.4, any DG A-module admits an Eilenberg-Moore resolution.However, it doesn't mean any DG A-module M with bounded below cohomology admits a minimal Eilenberg-Moore resolution (see Remark 6.3). Suppose that

$$\cdots \stackrel{\partial_{i+1}}{\to} H(A) \otimes V(i) \stackrel{\partial_i}{\to} \cdots \stackrel{\partial_2}{\to} H(A) \otimes V(1) \stackrel{\partial_1}{\to} H(A) \otimes V(0) \stackrel{\varepsilon}{\to} H(M) \to 0 \quad (4)$$

is a minimal free resolution of the graded $H(A)$-module $H(M)$. A natural question is when will M admit a minimal Eilenberg-Moore resolution induced from (4). The following proposition tells us some sufficient conditions.

Proposition 3.7. *The DG A-module M admits a minimal Eilenberg-Moore resolution, if either $\partial_A = \partial_M = 0$ or $\inf\{|v| \mid v \in V(i)\} > \sup\{|v| \mid v \in V(i-1)\}, i \geq 1$ is satisfied.*

Proof. If $\partial_A = 0$ and $\partial_M = 0$, then $H(A) = A^{\#}$ and $H(M) = M^{\#}$. Hence each free term $H(A) \otimes V(i)$ in (4) can be seen as a DG A-module with zero differential and the minimal free resolution (4) is in fact a complex of DG A-modules. By the construction procedure of Eilenberg-Moore resolution in [FHT, Proposition 20.11], we construct an Eilenberg-Moore resolution F of M as follows.

For each $V(i)$, set $\{e_{i,j} | j \in I_i\}$ as a basis of $V(i), i \geq 0$. Let $F(0) = A \otimes V(0)$ be a DG free A-module and define a DG morphism $f_0 : F(0) \to M$ by $f_0(e_{0,j}) = \varepsilon(e_{0,j})$. Since $H(A) = A^{\#}$, it is easy to see that $f_0 = \varepsilon$. Let $z_{1,j} = \partial_1(e_{1,j})$. We have $\varepsilon(z_{1,j}) = \varepsilon \circ \partial_1(e_{1,j}) = 0 \in H(M) = M^{\#}$. Surely, $f_0(z_{1,j}) = \varepsilon(z_{1,j}) = 0 = d_M(0)$. Define a semi-free A-module $F(1)$ such that $F(1)^{\#} = F(0)^{\#} \oplus (A^{\#} \otimes \Sigma V(1))$ and $\partial_{F(1)}(\Sigma e_{1,j}) = z_{1,j} = \partial_1(e_{1,j})$. Extend f_0 to $f_1 : F(1) \to M$ by $f_1(\Sigma e_{1,j}) = 0$.

Suppose inductively that we have constructed semi-free DG A-modules and DG morphisms: $F(0), f_0; F(1), f_1; \cdots ; F(n), f_n$ such that

$$F(l)^{\#} = A^{\#} \otimes (\bigoplus_{i=0}^{l} \Sigma^i V(i)),$$

$\partial_F(l)$ and f_l are defined by $\partial_{F(l)}(e_{0,j}) = \varepsilon(e_{0,j}), \quad \partial_{F(l)}(\Sigma^i e_{i,j}) = \Sigma^{i-1}\partial_i(e_{i,j})$; $f_l(e_{0,j}) = \varepsilon(e_{0,j}), f_l(e_{i,j}) = 0$, for any $i = 1, 2, \cdots, l$ and $l = 1, 2, \cdots, n$.

Let $Z_{n+1,j} = \partial_{n+1}(e_{n+1,j})$. It is easy to see that $\Sigma^n Z_{n+1,j}$ is a cocycle in $F(n)$ since $\partial_n \circ \partial_{n+1}(e_{n+1,j}) = 0$, and $f_n(\Sigma^n Z_{n+1,j}) = 0 = \partial_M(0) \in M$ by the induction hypothesis. Define a semi-free DG A-module F_{n+1} and a DG morphism $f_{n+1} : F(n+1) \to M$ such that $F_{n+1}^{\#} = F(n)^{\#} \oplus (A \otimes V(n+1))$, $\partial_{F(n+1)}|_{F(n)} = \partial_{F(n)}$, $\partial_{F(n+1)}(\Sigma^{n+1} e_{n+1,j}) = \Sigma^n Z_{n+1,j}$, $f_{n+1}|_{F(n)} = f_n$ and $f_{n+1}(\Sigma^{n+1} e_{n+1,j}) = 0$.

Then we have constructed $F(0), f_0; F(1), f_1; \cdots$, by induction. Let $F = \cup_{i\geq 0} F(i)$ and $f = \varinjlim f_i : F \to M$. By the construction procedure above and the assumption that $\partial_A = 0$, the constructed Eilenberg-Moore resolution F is just the cototaling complex

$$F = \coprod_{i=0}^{\infty} \Sigma^i A \otimes V(i)$$

of (4) with a differential defined by

$$\partial_F(x_0, \Sigma x_1, \cdots, \Sigma^n x_n, \cdots) = (\partial_1(x_1), \Sigma^1 \partial_2(x_2), \cdots, \Sigma^n \partial_{n+1}(x_{n+1}), \cdots).$$

It is easy to check that F is minimal since (4) is a minimal free resolution.

Next, we assume that $\inf\{|v| \,|\, v \in V(i)\} > \sup\{|v| \,|\, v \in V(i-1)\}, i \geq 1$ is satisfied. For any integer $i \geq 0$, let $\{e_{i_j} \,|\, j \in I_i\}$ be a basis of $V(i)$. Then the Eilenberg-Moore resolution F induced from the free resolution (4) admits a semi-basis $\bigcup_{i\geq 0}\{\Sigma^i e_{i_j} \,|\, j \in I_i\}$ according to the construction procedure in the proof of [FHT, Proposition 20.11]. For any semi-basis elements $\Sigma^i e^{i_j}$, we have

$$\partial_F(\Sigma^i e_{i_j}) = \sum_{l=0}^{i-1} \sum_{t\in I_l} a_{l_t} \Sigma^l e_{l_t} \qquad (5).$$

Since $\inf\{|v| \,|\, v \in V(i)\} > \sup\{|v| \,|\, v \in V(i-1)\}$ for any $i \geq 1$, we can conclude that each $a_{l_t} \in m_A$ by considering the degrees in (5). Hence $\partial(F) \subseteq m_A F$ and then F is minimal.

□

Remark 3.8. *Let A be a connected DG algebra. If either $\partial_A = 0$ or $H(A)$ is a p-Koszul graded algebra* (see [Ber], [HL]) *for some integer $p > 1$, then the DG A-module k satisfies the conditions in Proposition 3.7 and hence it admits a minimal Eilenberg-Moore resolution.*

Proposition 3.9. *Let M be a DG A-module such that $H(M)$ is bounded below. Then we have $\mathrm{grade}_{H(A)} H(M) \leq \mathrm{cl}_A M$. Furthermore, if $\mathrm{grade}_{H(A)} H(M) = \mathrm{cl}_A M$, then we have*

$$\mathrm{pd}_{H(A)} H(M) = \mathrm{grade}_{H(A)} H(M) = \mathrm{cl}_A M.$$

Proof. If $\mathrm{cl}_A M = +\infty$, then $\mathrm{grade}_{H(A)} H(M) \leq \mathrm{cl}_A M$ is obviously true and we have $\mathrm{pd}_{H(A)} H(M) = +\infty$ since $\mathrm{pd}_{H(A)} H(M) \geq \mathrm{cl}_A M$ by the existence of Eilenberg-Moore resolution. Hence it suffices to prove the proposition when $\mathrm{cl}_A M$ is finite.

Let $\mathrm{cl}_A M = n$. By the definition of cone length, M admits a semi-free resolution F_M with DGfree $\mathrm{class}_A F_M = n$. There exists a strictly increasing semi-free filtration of F_M: $0 = F_M(-1) \subset F_M(0) \subset F_M(1) \subset \cdots \subset F_M(n) = F_M$, where $F_M(i)/F_M(i-1) = A \otimes V(i), i \geq 0$. On the other hand, M admits an Eilenberg-Moore resolution F, which is constructed from a minimal free resolution:

$$\cdots \overset{\partial_{n+1}}{\to} H(A) \otimes W(n) \overset{\partial_n}{\to} \cdots \overset{\partial_1}{\to} H(A) \otimes W(0) \overset{\varepsilon}{\to} H(M) \to 0.$$

We denote $\{e_{j_i} | i \in I_j\}$ as a basis of $W(j), j \geq 0$. By [FHT, Proposition 6.6], there is a homotopy equivalence $f : F \to F_M$. Since $F_M(n) = F_M$, we have $f(F(r)) \subseteq F_M(r+n)$, for any $r \geq 0$. Hence there exists a minimal integer $p \leq n$ such that there is a DG morphism θ satisfying the conditions that $\theta \sim f$ and $\theta(F(r)) \subseteq F_M(r+p), r \geq 0$. We decompose θ as $\theta = \theta_p + \theta_{p-1} + \cdots + \theta_{-n}$, where

each $\theta_j(\Sigma^r W(r)) \subseteq A^{\#}\otimes V(r+j), r\geq 0$. Since $F_M^{\#}\cong \bigoplus\limits_{i=0}^{n} A^{\#}\otimes V(i)$, there is a minimal integer $t\leq n-p$ such that f is homotopic to a DG morphism θ such that $\theta(F(r))\subseteq F_M(p+r)$ and $\theta_p(\Sigma^j W(j))=0$, for any $j>t$. From $\partial_{F_M}\circ\theta=\theta\circ\partial_F$, we get that $\theta_P\circ d_0^F=d_0^{F_k}\circ\theta_P$, which implies that

$$\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t : (A\otimes W(t),\partial_A\otimes \mathrm{id})\to (A\otimes\Sigma^{-t-p}V(t+p),\partial_A\otimes\mathrm{id})$$

is a chain map. Then we get a morphism of $H(A)$-modules

$$[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t] : H(A)\otimes W(t)\to H(A)\otimes\Sigma^{-t-p}V(t+p).$$

Also by $\partial_{F_k}\circ\theta=\theta\circ\partial_F$, we have $\theta_p\circ d_1^F|_{\Sigma^{t+1}W(t+1)}=d_0^{F_k}\circ\theta_{p-1}|_{\Sigma^{t+1}W(t+1)}$. Hence the map $\mathrm{Hom}_{H(A)}(\partial_{t+1},H(A)\otimes\Sigma^{-t-p}V(t+p))$ maps $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]$ in $\mathrm{Hom}_{H(A)}(H(A)\otimes W(t),H(A)\otimes\Sigma^{-t-p}V(t+p))$ to 0, since

$$[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]\circ\partial_{t+1}=[\Sigma^{-t-p}\circ\theta_p\circ d_1^{F_k}\circ\Sigma^{t+1}]=[\Sigma^{-t-p}\circ d_0^{F_k}\circ\theta_{p-1}\circ\Sigma^{t+1}].$$

Thus $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]$ is a cocycle with cohomology degree t.

If $\mathrm{grade}_{H(A)}H(M)\geq n+1$, we want to get a contradiction. Suppose that $p\geq 0$. Then $\mathrm{grade}_{H(A)}H(M)=\inf\{i|\mathrm{Ext}^i_{H(A)}(H(M),H(A))\neq 0\}\geq n+1\geq p+t+1\geq t+1$. Hence $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]$ is a boundary. There is a graded $H(A)$-linear map $\psi: H(A)\otimes W(t-1)\to H(A)\otimes\Sigma^{-t-p}V(t+p)$ such that $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]=\psi\circ\partial_t$. The map ψ induces a DG morphism

$$\Psi : (A\otimes W(t-1),\partial_A\otimes\mathrm{id})\to (A\otimes\Sigma^{-t-p}V(t+p),\partial_A\otimes\mathrm{id})$$

such that $H(\Psi)=\psi$. For any $e_{(t-1)_i}$, there is $\lambda_i\in A\otimes\Sigma^{-t-p}V(t+p)$ such that $\psi([e_{(t-1)_i}])=[\lambda_i]$. Since $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]([e_{t_i}])=\psi\circ\partial_t([e_{t_i}])$, there is $x_i\in A\otimes\Sigma^{-t-p}V(t+p)$ such that

$$\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t(e_{t_i})-\Psi\circ\Sigma^{1-t}\circ d_1^{F_M}\circ\Sigma^t(e_{t_i})=\partial_A\otimes\mathrm{id}(x_i).$$

We define a homotopy map $h: F\to F_M$ by

$$h|_{A^{\#}\otimes\Sigma^{t-1}W(t-1)}=\Sigma^{t+p}\Psi\circ\Sigma^{1-t},\ h(e_{t_i})=\Sigma^{t+p}x_i$$

and $h|_{W_l}=0$, if $l\notin\{t-1,t\}$. Let $\theta'=\theta-h\circ\partial_F-\partial_{F_M}\circ h$. It is easy to check that $\theta'(\Sigma^r W(r))\subseteq F_M(r+p)$ and $\theta'_p(\Sigma^l W(l))=0$, for any $l\geq t-1$. This contradicts with the minimality of t. Hence $p\leq -1$ and f is homotopic to a DG morphism $f': F\to F_M$ such that $f'(F(r))\subseteq F_M(r-1), r\geq 0$. Therefore, $f'(F(0))=0$. On the other hand, e_{0_j} is a cocycle in F and each $[e_{0_j}]$ is non-zero in $H(F)$ by the construction of Eilenberg-Moore resolution. So $H(f')[e_{0_j}]=[f'(e_{0_j})]$ is non-zero, since f' is a homotopy equivalence. Then we reach a contradiction. Thus $\mathrm{grade}_{H(A)}H(M)\leq n$.

If $\mathrm{grade}_{H(A)}H(M)=n$, we claim that $p\leq 0$. Otherwise, $p\geq 1$. Then $\mathrm{grade}_{H(A)}H(M)=\inf\{i|\mathrm{Ext}^i_{H(A)}(H(M),H(A))\neq 0\}=n\geq p+t>t+1$. Hence $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]$ is a coboundary. There is a graded $H(A)$-linear map $\psi: H(A)\otimes W(t-1)\to H(A)\otimes\Sigma^{-t-p}V(t+p)$ such that $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]=\psi\circ\partial_t$. The map ψ induces a DG morphism

$$\Psi : (A\otimes W(t-1),\partial_A\otimes\mathrm{id})\to (A\otimes\Sigma^{-t-p}V(t+p),\partial_A\otimes\mathrm{id})$$

such that $H(\Psi)=\psi$. For any $e_{(t-1)_i}$, there is $\lambda_i\in A\otimes\Sigma^{-t-p}V(t+p)$ such that $\psi([e_{(t-1)_i}])=[\lambda_i]$. Since $[\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t]([e_{t_i}])=\psi\circ\partial_t([e_{t_i}])$, there is $x_i\in A\otimes\Sigma^{-t-p}V(t+p)$ such that

$$\Sigma^{-t-p}\circ\theta_p\circ\Sigma^t(e_{t_i})-\Psi\circ\Sigma^{1-t}\circ d_1^{F_k}\circ\Sigma^t(e_{t_i})=\partial_A\otimes\mathrm{id}(x_i).$$

We define a homotopy map $h: F\to F_M$ by

$$h|_{A^{\#}\otimes\Sigma^{t-1}W(t-1)}=\Sigma^{t+p}\Psi\circ\Sigma^{1-t},\ h(e_{t_i})=\Sigma^{t+p}x_i$$

and $h|_{W_l}=0$, if $l\notin\{t-1,t\}$. Let $\theta'=\theta-h\circ\partial_F-\partial_{F_k}\circ h$. It is easy to check that $\theta'(\Sigma^r W(r))\subseteq F_M(r+p)$ and $\theta'_p(\Sigma^l W(l))=0$, for any $l\geq t-1$. This contradicts with the minimality of t. Hence $p\leq 0$.

There exists a DG morphism $f':F\to F_M$ such that $f\sim f'$ and $f'(F(r))\subseteq F_M(r)$, for any $r\geq 0$. Let $g:F_M\to F$ be a homotopy inverse of f. By Proposition 2.11, g is homotopic to a DG morphism g' such that $g'(F_M(r))\subseteq F(r)$, for any $r\geq 0$. There is a DG morphism $g'':F_M\to F(n)$ such that $g'=\iota\circ g''$, where ι is the natural inclusion map: $F(n)\to F$. We have $\iota\circ g''\circ f'=g'\circ f'\sim g\circ f\sim \mathrm{id}_F$. Hence $E_1(\iota)\circ E_1(g'')\circ E_1(f')=E_1(\iota\circ g''\circ f')\sim \mathrm{id}$, since f',g'' and ι preserve the semi-free filtrations. We have $\mathrm{id}-E_1(\iota)\circ E_1(g''\circ f')=\partial\circ H+H\circ\partial$, where H is a homotopy map. Since the E_1 term of the Eilenberg-Moore spectral sequence induced from the semi-free filtration $0=F(-1)\subset F(0)\subset F(1)\cdots\subset F(i)\subset\cdots$ is

$$\cdots\stackrel{\partial_{i+1}}{\to} H(A)\otimes W(i)\stackrel{\partial_i}{\to}\cdots\stackrel{\partial_2}{\to} H(A)\otimes W(1)\stackrel{\partial_1}{\to} H(A)\otimes W(0)\to 0 \qquad (4),$$

the E_1 term of the Eilenberg-Moore spectral sequence induced from the semi-free filtration $0=F(-1)\subset F(0)\subset F(1)\cdots\subset F(n-1)\subset F(n)$ is

$$0\to H(A)\otimes_{A_0} W(n)\stackrel{\partial_n}{\to}\cdots\stackrel{\partial_2}{\to} H(A)\otimes W(1)\stackrel{\partial_1}{\to} H(A)\otimes W(0)\to 0.$$

Acting the functor $k\otimes_{H(A)}-$ on (4), we get a complex:

$$\cdots\to k\otimes W(i)\to k\otimes W(i-1)\to\cdots\to k\otimes W(0)\to 0,$$

where the differential from $k\otimes W(i)$ to $k\otimes W(i-1)$ is $k\otimes_{H(A)}\partial_i$, for any $i\geq 1$. Let $\widetilde{H}=k\otimes_{H(A)}H$. Then for any $i\geq n+1$, $\mathrm{id}|_{k\otimes W(i)}=\widetilde{H}\circ\widetilde{\partial}|_{k\otimes W(i)}+\widetilde{\partial}\circ\widetilde{H}|_{k\otimes W(i)}$ since $E_1(g''\circ f')(W(i))=0$. This implies that $\mathrm{Tor}_i^{H(A)}(k,H(M))=0, i\geq n+1$. Therefore, $\mathrm{pd}_{H(A)}H(M)=n$ since $\mathrm{pd}_{H(A)}H(M)\geq \mathrm{grade}_{H(A)}H(M)=n$. $\square$

4. GLOBAL DIMENSION OF DG ALGEBRAS

In ring theory and homological algebra, it is well known that the global dimension of a ring R is defined to be the supremum of the set of projective dimensions of all R-modules. Since the invariant cone length of a DG A-module plays a similar role in DG homological algebra as projective dimension of a module over a ring does in classic homological ring theory, it is natural for us to define global dimension of a DG algebra as follows.

Definition 4.1. *Let A be an connected DG algebra. The left global dimension and the right global dimension of A are respectively defined by*

$$l.\mathrm{Gl.dim}A=\sup\{\mathrm{cl}_A M|M\in \mathrm{D}(A)\} \text{ and } r.\mathrm{Gl.dim}A=\sup\{\mathrm{cl}_{A^{op}}M|M\in\mathrm{D}(A^{op})\}.$$

Remark 4.2. *Since $\mathrm{cl}_A M\leq \mathrm{pd}_{H(A)}H(M)$ for any DG A-module M, it is easy to see that $l.\mathrm{Gl.dim}A\leq \mathrm{gl.dim}H(A)$. For any connected DG algebra A, $l.\mathrm{Gl.dim}A=0$ if and only if $H(A)\simeq k$.*

Proposition 4.3. *Suppose that A is a connected DG algebra with zero differential. Then we have*

$$l.\mathrm{Gl.dim}A=\mathrm{gl.dim}A^{\#}=r.\mathrm{Gl.dim}A.$$

Proof. For any DG A-module M, $\mathrm{cl}_A M\leq \mathrm{pd}_{A^{\#}}H(M)$ by Remark 4.2. Hence $l.\mathrm{Gl.dim}A\leq \mathrm{gl.dim}A^{\#}$. On the other hand, k admits a minimal Eilenberg-Moore resolution by Remark 3.8. Applying Theorem 3.6 on k, we get that $\mathrm{cl}_A k=\mathrm{pd}_{A^{\#}}k=\mathrm{gl.dim}A^{\#}$. Hence $l.\mathrm{Gl.dim}A=\mathrm{gl.dim}A^{\#}$. Similarly, we can prove that $r.\mathrm{Gl.dim}A=\mathrm{gl.dim}A^{\#}$ since $\mathrm{pd}_{A^{\#}}k=\mathrm{pd}_{A^{op\#}}k=\mathrm{gl.dim}A^{\#}$. $\square$

The proposition above indicates that the global dimension of a DG algebra with zero differential coincides with the global dimension of its underlying graded algebra. Therefore, Definition 4.1 can be seen as a generalization of the classical definition of global dimension for graded algebras. It is natural for us to study the connected DG algebras with small global dimension.

Proposition 4.4. *Let A be a connected DG algebra. Then we have*
$$\mathrm{cl}_A k = 1 \Leftrightarrow l.\mathrm{Gl.dim}A = 1 \Leftrightarrow \mathrm{gl.dim}H(A) = 1.$$

Proof. We first prove that $\mathrm{cl}_A k = 1 \Rightarrow \mathrm{gl.dim}H(A) = 1$. By Theorem 3.5, ${}_A k$ admits a minimal semi-free resolution F_k with DGfree $\mathrm{class}_A F_k = 1$. The semi-free module F_k admits a strictly increasing semi-free filtration:
$$0 \subset F_k(0) \subset F_k(1) = F_k,$$
where $F_k(0)$ is a DG free A-submodule of F_k. We claim that the DG free A-module $F_k(0)$ admits a cocycle basis concentrated in degree 0. Indeed, for any DG free direct summand Ae of $F_k(0)$, e is a cocycle element in F_k. By Remark 1.3, F_k is concentrated in degrees ≥ 0. Hence $|e| \geq 0$. If $|e| \geq 1$, then e is a boundary of F_k since $H(F_k) = k$. There is $x \in F_k$ such that $d(x) = e$. This contradicts with the minimality of F_k. Hence $|e| = 0$ and $F_k(0)$ admits a cocycle basis concentrated in degree 0. By the linearly split short exact sequence: $0 \to F_k(0) \xrightarrow{i} F_k \xrightarrow{p} F_k/F_k(0) \to 0$, we can get a long exact sequence of cohomologies:
$$0 \to H^0(F_k(0)) \xrightarrow{H^0(i)} k \xrightarrow{H^0(p)} H^0(F_k/F_k(0)) \xrightarrow{\delta^0} H^1(F_k(0)) \to 0 \to \cdots$$
$$\cdots \to 0 \to H^n(F_k/F_k(0)) \xrightarrow{\delta^n} H^{n+1}(F_k(0)) \to 0 \cdots.$$

From this long exact sequence, we conclude that $H^0(i)$ is a monomorphism. Hence the basis of $F_k(0)$ contains only one element. Let $F_k(0) = Ae^0$. It is easy to check that $H^0(i)$ and each $\delta^j, j \geq 0$ are isomorphism. We have $\ker H(i) \cong \oplus_{j\geq 0} H^{j+1}(F_k(0))$. Let $F_k/F_k(0) = \oplus_{j\in I_1} Ae_j$. By the definition of connecting homomorphism, we have $[a]\delta^{|e_j|}([e_j]) = \delta^{|a|+|e_j|}([ae_j])$, for any $j \in I_1$ and any cocycle $a \in A$. Hence, we have a morphism of graded $H(A)$-modules:
$$\partial_1 : H(\Sigma^{-1}(F_k/F_k(0))) \to H(F_k(0))$$
defined by $\partial_1([\Sigma^{-1}e_j]) = \delta^{|e_j|}[e_j]$. Let $\sum_{j\in I_1}[a_j][\Sigma^{-1}e_j] \in (\mathrm{Ker}(\partial_1))^n, n \geq 0$. Then we have $\sum_{j\in I_1}[a_j]\delta^{|e_j|}([e_j]) = \delta^n(\sum_{j\in I_1}[a_j][e_j]) = 0$. Since $\delta^n, n \geq 0$ is an isomorphism, we have $(\sum_{j\in I_1}[a_j][e_j]) = 0$. This implies that $[a_j] = 0, j \in I_1$, so ∂_1 is a monomorphism. Let $[be^0] \in H^n(F_k(0)), n \geq 1$. There exists $\sum_{j\in I_1}[a_j][e_j] \in H^n(F_k/F_k(0))$ such that $[be^0] = \delta^n(\sum_{j\in I_1}[a_j][e_j]) = \partial_1(\sum_{j\in I_1}[a_j][\Sigma^{-1}e_j])$ since δ^n is an isomorphism. Therefore, $\mathrm{Im}(\partial_1) = \oplus_{j\geq 0} H^{j+1}(F_k(0))$. Then we can get a minimal free resolution
$$0 \to H(\Sigma^{-1}(F_k/F_k(0))) \xrightarrow{\partial_1} H(F_k(0)) \xrightarrow{H(i)} k \to 0$$
of the $H(A)$-module k. Hence $\mathrm{gl.dim}H(A) = 1$.

Next we prove that $\mathrm{gl.dim}H(A) = 1 \Rightarrow l.\mathrm{Gl.dim}A = 1$. By Remark 4.2, we have $l.\mathrm{Gl.dim}A \leq \mathrm{gl.dim}H(A) = 1$. If $l.\mathrm{Gl.dim}A = 0$, then $\mathrm{cl}_A k = 0$. This implies that $A \simeq k$. And so $H(A) \cong k$, which contradicts with $\mathrm{gl.dim}H(A) = 1$. Hence $\mathrm{cl}_A k = 1$.

Finally, we prove that $l.\mathrm{Gl.dim}A = 1 \Rightarrow \mathrm{cl}_A k = 1$. By the definition of $l.\mathrm{Gl.dim}A$, we have $\mathrm{cl}_A k \leq 1$. If $\mathrm{cl}_A k = 0$, then it is easy to check that $A \simeq k$ and hence $l.\mathrm{Gl.dim}A = 0$. So $\mathrm{cl}_A k = 1$.

□

The proposition above tells us a characterization of the connected DG algebras with global dimension 1. The connected DG algebras with global dimension 2 generally don't admit the similar properties (see Example 6.1). However, we have the following proposition.

Proposition 4.5. *Let A be a connected DG algebra with* $\mathrm{gl.dim}H(A) = 2$*, then*

$$l.\mathrm{Gl.dim}A = \mathrm{cl}_A k = 2.$$

Proof. Since $\mathrm{gl.dim}H(A) = 2$, we have $\mathrm{cl}_A k \leq l.\mathrm{Gl.dim}A \leq 2$ by Remark 4.2. It suffice to prove that $\mathrm{cl}_A k = 2$. By Proposition 4.4, we have $\mathrm{cl}_A k \neq 1$ since $\mathrm{gl.dim}H(A) \neq 1$. By Remark 4.2, we have $\mathrm{cl}_A k \neq 0$ since $H(A) \not\cong k$. Therefore, $\mathrm{cl}_A k = 2$. □

The converse of Proposition 4.5 is generally not true. Example 6.1 can serve as a counter example (see Remark 6.3). The following interesting proposition tells us the close relations between $l.\mathrm{Gl.dim}A$ and $\mathrm{gl.dim}H(A)$ under some special assumptions.

Proposition 4.6. *Let A be a connected DG algebra. If either* $\mathrm{cl}_A k$ *or* $\mathrm{gl.dim}H(A)$ *is finite and equals to* $\mathrm{depth}_{H(A)}H(A)$*, then*

$$l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k.$$

Proof. If $\mathrm{cl}_A k = \mathrm{depth}_{H(A)}H(A) < +\infty$, then by Proposition 3.9 and the fact that $\mathrm{gl.dim}H(A) = \mathrm{pd}_{H(A)}k$ and $\mathrm{depth}_{H(A)}H(A) = \mathrm{grade}_{H(A)}k$, we have

$$\mathrm{depth}_{H(A)}H(A) = \mathrm{cl}_A k = \mathrm{pd}_{H(A)}k = \mathrm{gl.dim}H(A).$$

Hence $l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k$ by Remark 4.2.

If $\mathrm{gl.dim}H(A) = \mathrm{depth}_{H(A)}H(A) < +\infty$, then by Remark 4.2, we have

$$\mathrm{cl}_A k \leq l.\mathrm{Gl.dim}A \leq \mathrm{gl.dim}H(A) + \infty.$$

Hence $\mathrm{cl}_A k \geq \mathrm{grade}_{H(A)}k = \mathrm{depth}_{H(A)}H(A)$ by Proposition 3.9. Therefore

$$l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k$$

since $\mathrm{gl.dim}H(A) = \mathrm{depth}_{H(A)}H(A)$ by the assumption. □

Remark 4.7. *Suppose that A is a connected DG algebra such that $H(A)$ is an Artin-Schelter regular algebra. Then by Proposition 4.6, we have*

$$l.\mathrm{Gl.dim}A = \mathrm{gl.dim}H(A) = \mathrm{cl}_A k.$$

Note that Proposition 4.5 is not a direct corollary of Theorem 4.6. For example, let A be a connected DG algebra such that the graded algebra

$$H(A) \cong k\langle x, y\rangle/(xy).$$

Then by Proposition 4.5, we have $l.\mathrm{Gl.dim}A = \mathrm{cl}_A k = 2$*. This can't be concluded by Theorem 4.6, since* $\mathrm{gl.dim}H(A) = 2 \neq \mathrm{depth}_{H(A)}H(A) = 1$.

5. Characterizations of homologically smooth DG algebras

In DG homological algebra, homologically smooth DG algebras play a similar role as regular ring do in classical homological ring theory. Just as regular rings can be characterized by homological properties of their modules, homologically smooth DG algebras can be studied through their DG modules. In this section, we give some characterizations of homologically smooth DG algebras.

Proposition 5.1. *Let A be a connected DG algebra with such that* $\mathrm{cl}_{A^e} A < +\infty$*. Then for any DG A-module M, we have* $\mathrm{cl}_A M \leq \mathrm{cl}_{A^e} A < +\infty$.

Proof. Let $\mathrm{cl}_{A^e}A = n$. By the definition of cone length, the DG A^e-module A admits a semi-free resolution X such that DGfree $\mathrm{class}_{A^e}X = n$. This implies that X admits a strictly increasing semi-free filtration

$$0 \subset X(0) \subset X(1) \subset \cdots \subset X(n) = X,$$

where $X(0) = A^e \otimes V(0)$ and each $X(i)/X(i-1) \cong A^e \otimes V(i), i \geq 1$, is a DG free A^e-module. Let $E_i = \{e_{i_j} | j \in I_i\}, i \geq 0$, be a basis of $V(i)$. For any $i \geq 1$, define $f_i : A^e \otimes_k \Sigma^{-1}V(i) \to X(i-1)$ such that $f_i(\Sigma^{-1}e^{i_j}) = \partial_{X(i)}(e^{i_j})$. By Lemma 3.3, $X(i) \cong \mathrm{cone}(f_i), i = 1, 2, \cdots, n$.

For any DG A-module M, let F_M be its semi-free resolution. As a DG A-module, $X(i) \otimes_A F_M \cong \mathrm{cone}(f_i \otimes_A \mathrm{id}_{F_M})$, $i = 1, 2, \cdots, n$. Since $A^e \otimes_A F_M \cong A \otimes_k F_M$, we have

$$(A^e \otimes V(i)) \otimes_A F_M \cong A \otimes_k V(i) \otimes_k F_M, \quad i = 0, 1, \cdots, n.$$

Choose a subset $\{m\} \subseteq F_M$ such that each m is a cocycle and $\{[m]\}$ is a basis of the k-vector space $H(F_M)$. Define a DG morphism

$$\phi_i : A \otimes_k V(i) \otimes_k H(F_M) \to A \otimes_k V(i) \otimes_k F_M$$

such that $\phi_i(a \otimes v \otimes [m]) = a \otimes v \otimes m$, for any $a \in A, v \in V(i)$ and $[m]$. It is easy to check that ϕ_i is a quasi-isomorphism.

In the following, we prove inductively that $\mathrm{cl}_A(X(i) \otimes_A F_M) \leq i, i = 0, 1, \cdots, n$. Since $\phi_0 : A \otimes V(0) \otimes H(F_M) \to X(0) \otimes_A F_M$ is a quasi-isomorphism, we have $\mathrm{cl}_A(X(0) \otimes_A F_M) = 0$. Suppose inductively that we have proved that

$$\mathrm{cl}_A(X(l) \otimes_A F_M) \leq l, \, l \geq 0.$$

We should prove $\mathrm{cl}_A(X(l+1) \otimes_A F_M) \leq l+1$. Since $\mathrm{cl}_A(X(l) \otimes_A F_M) \leq l$, there is a semi-free resolution $\varphi_l : F_l \xrightarrow{\simeq} X(l) \otimes_A F_M$ such that DGfree $\mathrm{class}_A F_l \leq l$. Because $A \otimes_k \Sigma^{-1}V(l+1) \otimes_k H(F_M)$ is semi-free, there is a DG morphism

$$\psi_l : A \otimes_k \Sigma^{-1}V(l+1) \otimes_k H(F_M) \to F_l$$

such that $\varphi_l \circ \psi_l \sim (f_l \otimes_A \mathrm{id}_{F_M}) \circ \phi_{l+1}$.

For convenience, we write $G(l+1) = A \otimes_k V(l+1)$ and $K(l+1) = A^e \otimes V(l+1)$. In $\mathrm{D}(A)$, there is a DG morphism $h_{l+1} : \mathrm{cone}(\psi_l) \to X(l+1) \otimes_A F_M$ making the diagram

$$\begin{tikzcd}
\Sigma^{-1}G(l+1) \otimes_k H(F_M) \arrow[r, "\psi_l"] \arrow[d, "\Sigma^{-1}(\phi_{l+1})"] & F_l \arrow[r, "\lambda_l"] \arrow[d, "\varphi_l"] & \mathrm{cone}(\psi_l) \arrow[r, "\eta_l"] \arrow[d, "\exists h_{l+1}"] & G(l+1) \otimes_k H(F_M) \arrow[d, "\phi_{l+1}"] \\
\Sigma^{-1}K(l+1) \otimes_A F_M \arrow[r, "f_l \otimes_A \mathrm{id}_{F_M}"] & X(l) \otimes_A F_M \arrow[r, "\iota_l"] & X(l+1) \otimes_A F_M \arrow[r, "\pi_l"] & K(l+1) \otimes_A F_M
\end{tikzcd}$$

commute. By five-lemma, h_{l+1} is an isomorphism in $\mathrm{D}(A)$. This implies that there are quasi-isomorphisms $g : Y \to \mathrm{cone}(\psi_l)$ and $t : Y \to X(l+1) \otimes_A F_M$, where Y is some DG A-module. Hence $\mathrm{cl}_A(X(l+1) \otimes_A F_M) = \mathrm{cl}_A Y = \mathrm{cl}_A \mathrm{cone}(\psi_l) \leq l+1$.

We have proved inductively that $\mathrm{cl}_A(X \otimes_A F_M) \leq n$. Since $F_M \simeq X \otimes_A F_M$, we get $\mathrm{cl}_A M \leq n$.

$\square$

Remark 5.2. *The proposition above indicates that* $\mathrm{cl}_{A^e}A$ *is an upper bound of the* $l.\mathrm{Gl.dim}A$. *Suppose that* A *is a homologically smooth DG algebra. Then* A *admits finite global dimension since* ${}_{A^e}A$ *is compact and therefore* $\mathrm{cl}_{A^e}A$ *is finite.*

Corollary 5.3. *Let* A *be a connected DG algebra with* $\mathrm{cl}_{A^e}A = \mathrm{cl}_A M$ *for some DG* A*-module* M. *Then* $l.\mathrm{Gl.dim}A = \mathrm{cl}_{A^e}A$.

Proof. If $\mathrm{cl}_A M = +\infty$, then $l.\mathrm{Gl.dim}A = \sup\{\mathrm{cl}_A M | M \in \mathrm{D}(A)\} = +\infty$. Hence we only need to consider the case that $\mathrm{cl}_A M < +\infty$. Since $\mathrm{cl}_{A^e} A = \mathrm{cl}_A M$, we have $\mathrm{cl}_{A^e} A < +\infty$. Then $l.\mathrm{Gl.dim}A \le \mathrm{cl}_{A^e} A = \mathrm{cl}_A M$ by Proposition 5.1. On the other hand, $l.\mathrm{Gl.dim}A = \sup\{\mathrm{cl}_A M | M \in \mathrm{D}(A)\} \ge \mathrm{cl}_A M$. So $l.\mathrm{Gl.dim}A = \mathrm{cl}_A M = \mathrm{cl}_{A^e} A$. $\square$

It is well known that a complex in the bounded derived category of finitely generated modules over a Noetherian ring is perfect if and only if it has finite projective dimension. The following proposition indicates that we have a similar result in DG context.

Proposition 5.4. *Let A be a connected cochain DG algebra such that $H(A)$ is a Noetherian graded algebra. Then for any object M in $\mathrm{D}^f(A)$, M is compact if and only if $\mathrm{cl}_A M < +\infty$.*

Proof. If M is compact, then M admits a minimal semi-free resolution F_M with a finite semi-basis. By Theorem 3.5, $\mathrm{cl}_A M = \mathrm{DGfree\ class}_A F_M < +\infty$.

Conversely, if $\mathrm{cl}_A M < +\infty$, we want to prove that M is compact. Set $\mathrm{cl}_A M = n$. By Theorem 3.5, there exists a minimal semi-free resolution F_M of M such that $\mathrm{DGfree.class}_A F_M = n$. There is a semi-free filtration

$$0 = F_M(-1) \subset F_M(0) \subset F_M(1) \subset \cdots \subset F_M(n) = F_M$$

of F_M. Each $F_M(i)/F_M(i-1) = A \otimes V(i)$ is a DG free A-module on a cocycle basis, $i = 0, 1, \cdots, n$. We should prove that each $V(i)$ is finite dimensional.

Let $\{e_{i,j} | j \in I_i\}$ be a basis of $V(i), i = 0, 1, \cdots, n$. Let $\iota_0 : F_M(0) \to F_M$ be the inclusion map. Since $\mathrm{im}H(\iota_0)$ is a graded $H(A)$-submodule of $H(F_M)$ and $H(A)$ is Noetherian, $\mathrm{im}H(\iota_0) \cong \frac{H(F_M(0))}{\ker H(\iota_0)}$ is also a finitely generated $H(A)$-module. Let $\mathrm{im}H(\iota_0) = H(A)f_{0,1} + H(A)f_{0,2} + \cdots + H(A)f_{0,n}$. Since $H(F_M(0)) \cong \bigoplus\limits_{j \in I_0} H(A)e_{0,j}$ is a free graded $H(A)$-module, there is a finite subset $J_0 = \{i_1, i_2, \cdots, i_l\}$ of I_0 such that

$$f_{0,s} = \sum_{t=1}^{l} a_{st}\overline{e_{0,i_t}}, s = 1, 2, \cdots, n,$$

where each $a_{st} \in H(A)$. If $V(0)$ is infinite dimensional, then both I_0 and $I_0 \setminus J_0$ are infinite sets. Hence for any $j \in I_0 \setminus J_0$, we have $e_{0,j} \in \ker H(\iota_0)$. Since $[\iota_0(e_{0,j})] = [e_{0,j}] = 0$ in $H(F_M)$, there exist $x_{o,j} \in F$ such that $\partial_{F_M}(x_{0,j}) = e_{0,j}$. This contradict with the minimality of F_M. Thus $V(0)$ is finite dimensional and $F_M(0) \in \mathrm{D}^f(A)$.

Suppose inductively that we have proved that $V(0), \cdots, V(i-1)$ are finite dimensional. Then each $F_M(j)/F_M(j-1), (j = 0, 1, \cdots, i-1)$ is an object in $D^f(A)$. And we can prove inductively that each $F(j), (j = 0, 1, \cdots, i-1)$ is in $\mathrm{D}^f(A)$ by the following sequence of short exact sequences

$$0 \longrightarrow F_M(j-1) \longrightarrow F_M(j) \longrightarrow F_M(j)/F_M(j-1) \longrightarrow 0, j = 1, \cdots, i-1.$$

Similarly, $F_M/F_M(i-1)$ is also an object in $\mathrm{D}^f(A)$ by the short exact sequence

$$0 \longrightarrow F_M(i-1) \longrightarrow F_M \longrightarrow F_M/F_M(i-1) \longrightarrow 0.$$

On the other hand, it is easy to see that $F_M/F_M(i-1)$ is also a minimal semi-free DG A-module and it has a semi-free filtration

$$F_M(i)/F_M(i-1) \subseteq F_M(i+1)/F_M(i-1) \subseteq \cdots \subseteq F_M(n)/F_M(i-1) = F_M/F_M(i-1).$$

Let $\iota_i : F_M(i)/F_M(i-1) \to F_M/F_M(i-1)$ be the inclusion morphism. Since $\mathrm{im}H(\iota_i)$ is a graded $H(A)$-submodule of $H(F_M/F_M(i-1))$ and $H(A)$ is Noetherian, $\mathrm{im}H(\iota_i) \cong \frac{H(F_M(i)/F_M(i-1))}{\ker H(\iota_i)}$ is also a finitely generated $H(A)$-module. Let $\mathrm{im}H(\iota_i) = H(A)f_{i,1} + H(A)f_{i,2} + \cdots + H(A)f_{i,m}$. Since

$$H(F_M(i)/F_M(i-1)) \cong \bigoplus_{j\in I_i} H(A)e_{i,j}$$

is a free graded $H(A)$-module, there is a finite subset $J_i = \{s_1, s_2, \cdots, s_r\}$ of I_i such that

$$f_{i,l} = \sum_{t=1}^{r} a_{l,t}\overline{e_{i,s_t}}, l = 1, 2, \cdots, m,$$

where each $a_{lt} \in H(A)$. If $V(i)$ is an infinite dimensional space, then both I_i and $I_i \setminus J_i$ are infinite sets. Hence for any $j \in I_i \setminus J_i$, we have $e_{i,j} \in \ker H(\iota_i)$. Since $[\iota_i(e_{i,j})] = [e_{i,j}] = 0$ in $H(F_M/F_M(i-1))$, there exist $x_{i,j} \in F_M/F_M(i-1)$ such that $\partial_{F_M}(x_{i,j}) = e_{i,j}$. This contradict with the minimality of F_M. Thus $V(i)$ is finite dimensional.

By the induction above, we prove that each $V(i), (i = 0, 1, \cdots, n)$ is finite dimensional. Hence F_M has a finite semi-basis and M is compact.

□

Assume that A is a connected DG algebra such that $H(A)$ is a left Noetherian graded algebra with finite global dimension. It is easy to prove that $\mathrm{D}^f(A) = \mathrm{D}^c(A)$ (see [MW2, Corollary 3.7]) by using the existence of Eilenberg-Moore resolution. Both Example 6.1 in the last section and [MW2, Example 3.12] show that the condition that $H(A)$ is a left Noetherian graded algebra with finite global dimension is much stronger than the condition that A is homologically smooth. A natural question is whether $\mathrm{D}^f(A) = \mathrm{D}^c(A)$ is still right under the latter weaker condition. We have the following theorem.

Theorem 5.5. *Let A be a connected cochain DG algebra such that $H(A)$ is a Noetherian graded algebra. Then the following are equivalent:*
(a) *A is homologically smooth;*
(b) $\mathrm{cl}_A k < +\infty$*;*
(c) $\mathrm{cl}_{A^e} A < +\infty$*;*
(d) $\mathrm{D}^c(A) = \mathrm{D}^f(A)$*;*
(e) $l.\mathrm{Gl.dim}A < +\infty$*.*

Proof. (a) ⇒ (b) By Remark 1.6, $k \in \mathrm{D}^c(A)$ since A is homologically smooth. This implies that $\mathrm{cl}_A k < +\infty$ by Proposition 5.4.

(b) ⇒ (c) By Proposition 5.4, $k \in \mathrm{D}^c(A)$ since $\mathrm{cl}_A k < +\infty$. By Remark 1.6, $A \in \mathrm{D}^c(A^e)$. Then the DG A^e-module A admits a minimal semi-free resolution X with a finite semi-basis. Since X has a finite semi-basis, its DG free class is finite. By the definition of cone length of a DG module,

$$\mathrm{cl}_{A^e} A \le \text{DGfree class}_{A^e} X < +\infty.$$

(c) ⇒ (d) It suffices to prove that any object M in $\mathrm{D}^f(A)$ is compact since any compact DG A-module is obviously in $\mathrm{D}^f(A)$. By Proposition 5.1,

$$\mathrm{cl}_A M \le \mathrm{cl}_{A^e} A < +\infty.$$

By Proposition 5.4, $M \in \mathrm{D}^c(A)$.

(d) ⇒ (a) By the assumption that $\mathrm{D}^c(A) = \mathrm{D}^f(A)$, we have $k \in \mathrm{D}^c(A)$. Hence A is homologically smooth by Remark 1.6.

(e) ⇒ (b) By the definition of global dimension of DG algebras, we have

$$\mathrm{cl}_A k \le l.\mathrm{Gl.dim}A < +\infty.$$

$(c) \Rightarrow (e)$ By Proposition 5.1, $l.\mathrm{Gl.dim} A \leq \mathrm{cl}_{A^e} A < +\infty$.

□

6. A counter example

Suppose that A is a connected cochain DG algebra such that $H(A)$ is a left Noetherian graded algebra. By the existence of Eilenberg-moore resolution, any object in $\mathrm{D}^f(A)$ is compact if $\mathrm{gl.dim} H(A) < \infty$. A natural question is whether the converse is right. In this section, we will give a counter example to show that it is generally not true.

Example 6.1. *Let A be a connected DG algebra such that $A^{\#} = k\langle x, y\rangle$ with $|x| = |y| = 1$ and its differential ∂_A is defined by $\partial_A(x) = y^2$ and $\partial_A(y) = 0$.*

Proposition 6.2. *Let A be the connected DG algebra in Example 6.1. Then A is a Koszul, homologically smooth DG algebra with $l.\mathrm{Gl.dim} A = 2$ but*

$$H(A) \cong k[y, xy + yx]/(y^2).$$

Proof. Using the constructing procedure in [MW1, Proposition 2.4], we can construct a minimal semi-free resolution F of k with

$$F^{\#} = A^{\#} \oplus A^{\#} \cdot \Sigma e_y \oplus A^{\#} \cdot \Sigma e_z,$$

where $z = x + y\Sigma e_y$, and ∂_F is defined by $\partial_F(\Sigma e_y) = y, \quad \partial_F(\Sigma e_z) = z$. And the quasi-isomorphism between F and k is $\theta : F \to k$ defined by

$$\theta(a + a_y \Sigma e_y + a_z \Sigma e_z) = \varepsilon(a).$$

Since the minimal semi-free resolution F of k has a finite semi-basis $\{1, \Sigma e_y, \Sigma e_z\}$ concentrated in degree 0. Therefore, A is a Koszul, homologically smooth DG algebra. It is easy to see that F admits a strictly increasing semi-free filtration

$$0 \subset F(0) \subset F(1) \subset F(2) = F,$$

where $F(0) = A, F(1)/F(0) = A \cdot \Sigma e_y$ and $F(2)/F(1) = A \cdot \Sigma e_z$ are all DG free A-modules. Hence $\mathrm{cl}_A k = \mathrm{DGfree.class}_A F \leq 2$.

By a straight forward computation, we get $H(A) \cong k[y, xy + yx]/(y^2)$ and $\mathrm{depth}_{H(A)} H(A) = 1$. Since $A \not\simeq k$, we have $\mathrm{cl}_A k \neq 0$. We can also conclude by Theorem 3.9 that $\mathrm{cl}_A k \neq 1$ since $\mathrm{depth}_{H(A)} H(A) = 1$ and $\mathrm{gl.dim}\, H(A)$ is not finite. Hence $\mathrm{cl}_A k = 2$.

Using the constructing procedure in [MW1, Proposition 2.4], we can construct a minimal semi-free resolution X of the DG A^e-module A with

$$X^{\#} = (A^e)^{\#} \oplus (A^e)^{\#} \cdot \Sigma e_z \oplus (A^e)^{\#} \cdot \Sigma e_t$$

where

$$z = y \otimes 1 - 1 \otimes y, \quad t = (y \otimes 1 - 1 \otimes y) \cdot \Sigma e_z + (x \otimes 1 - 1 \otimes x),$$

and the differential of X is defined by $\partial_X(\Sigma e_z) = z$ and $\partial_X(\Sigma e_t) = t$. The quasi-isomorphism between X and A is $\theta : X \to A$ defined by

$$\theta(a \otimes b + (a_1 \otimes b_1)\Sigma e_z + (a_2 \otimes b_2)\Sigma e_t) = ab.$$

It is easy to check that X admits a semi-free filtration

$$0 \subset X(0) \subset X(1) \subset X(2) = X,$$

where $X(0) = A^e, X(1)/X(0) = A^e \cdot \Sigma e_y$ and $X(2)/X(1) = A^e \cdot \Sigma e_z$ are all DG free A^e-modules. Hence

$$\mathrm{cl}_{A^e} A = \mathrm{DGfree.class}_{A^e} X \leq 2.$$

By Proposition 5.1, we have $\mathrm{cl}_{A^e} A \geq \mathrm{cl}_A k = 2$. Therefore $\mathrm{cl}_{A^e} A = \mathrm{cl}_A k = 2$ and we have $l.\mathrm{Gl.dim} A = 2$ by Corollary 5.3. □

Remark 6.3. *By Proposition 6.2, the DG algebra A in Example 6.1 is a homologically smooth DG algebra with* $l.\mathrm{Gl.dim}A = 2$ *and* $H(A) \cong k[y, xy + yx]/(y^2)$*. Hence any DG A-module in* $\mathrm{D}^f(A)$ *is compact by Theorem 5.5, while $H(A)$ is a Noetherian graded algebra with* $\mathrm{gl.dim}H(A) = +\infty$*. Example 6.1 also indicates that the converse of Proposition 4.5 is generally not true. Moreover, the DG A-module k in Example 6.1 doesn't admit a minimal Eilenberg-Moore resolution by Proposition 3.6 since* $\mathrm{cl}_A k = 2 \neq \mathrm{pd}_{H(A)} k = \mathrm{gl.dim}H(A) = +\infty$.

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